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# Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates

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**ABSTRACT**

We characterize a three-factor model of commodity spot prices, convenience yields, and interest rates, which nests many existing specifications. The model allows convenience yields to depend on spot prices and interest rates. It also allows for time-varying risk premia. Both may induce mean reversion in spot prices, albeit with very different economic implications. Empirical results show strong evidence for spot-price level dependence in convenience yields for crude oil and copper, which implies mean reversion in prices under the risk-neutral measure. Silver, gold, and copper exhibit time variation in risk premia that implies mean reversion of prices under the physical measure.

COMMODITY DERIVATIVES MARKETS HAVE WITNESSED tremendous growth in recent years. A variety of models have been proposed for pricing commodity derivatives such as futures and options.[1] In his presidential address, Schwartz (1997) selects and empirically compares three models. His empirical results suggest that three factors, spot prices, interest rates, and convenience yields, are necessary to capture the dynamics of futures prices. Further, models that accommodate mean reversion in spot prices under the risk-neutral measure seem desirable, although in that case, Schwartz argues that commodities cannot be seen as "an asset in the usual sense,"[2] because they do not satisfy the standard no-arbitrage condition for traded assets.

Below, we develop a three-factor Gaussian model of commodity futures prices, which nests the three specifications analyzed by Schwartz (1997) as well as Brennan (1991), Gibson and Schwartz (1990), Ross (1997), and Schwartz and Smith (2000). Instead of modeling separately the dynamics of spot prices, interest rates, and the convenience yield process, we start by directly specifying

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[1] An early description of the market can be found in the collection of papers in Jameson (1995). Seppi (2003) offers a more up-to-date survey.

[2] See, for example, Schwartz (1997, p. 926), and Ross (1997).

the most general identifiable three- (latent-) factor Gaussian model of futures prices.[3] Assuming the term structure of risk-free interest rates is driven by a single factor, and imposing a restriction on the drift of spot prices that amounts to the standard no-arbitrage condition, we identify the convenience yield implied by our general model of futures prices. We show that it allows for a richer unconditional covariance structure of convenience yields, commodity prices, and interest rates than previous models. In particular, the convenience yield may depend both on the spot price and the risk-free rate itself. This model is "maximal" in the sense of Dai and Singleton (DS) (2000).

One simple insight of our framework is that the models by Ross (1997) and Schwartz (1997), which allow for mean reversion under the risk-neutral measure of spot prices, can be simply interpreted as arbitrage-free models of commodity spot prices, where the convenience yield is a function of the spot price. Spot price level dependence in convenience yields leads to mean reversion of spot prices under the risk-neutral measure. The latter feature seems to be empirically desirable to fit the cross section of futures prices.

Several papers (Working (1949), Brennan (1958), Deaton and Laroque (1992), Routledge, Seppi, and Spatt (RSS) (2000) have shown that convenience yields arise endogenously as a result of the interaction among supply, demand, and storage decisions. In particular, RSS (2000) show that in a competitive rational expectations model of storage, when storage in the economy is driven to its lower bound, for example in periods of relative scarcity of the commodity available for trading, convenience yields should be high. This provides some economic rationale for allowing the convenience yield to depend on spot prices as in our model. Indeed, assuming that periods of scarcity, for example, low inventory, correspond to high spot prices, this theory predicts a positive relation between the convenience yield and spot prices.

Further, RSS (2000) note that the correlation structure between spot prices and convenience yields should be time varying, in contrast to the prediction of standard commodity derivatives pricing models such as Brennan (1991), Gibson and Schwartz (1990), Amin, Ng, and Pirrong (1995), Schwartz (1997), and Hilliard and Reis (1998). While the model we develop has a constant instantaneous correlation structure (since it is Gaussian), it allows for a more general unconditional correlation structure of spot price and convenience yields than previous papers.

Most theoretical models of convenience yields (such as RSS (2000)) assume that interest rates are 0, and thus do not deliver predictions about how interest rates should affect the convenience yield. However, to the extent that inventory and interest rates are correlated, finding a relation between interest rates and convenience yields seems to be consistent with the theory. Further, interest rates in general proxy (at least partially) for economic activity, which in turn may affect convenience yields.

To empirically implement the model and estimate the significance of the previously imposed overidentifying restrictions, we need a specification of risk

[3] In Dai and Singleton's (2000) terminology, we use the "maximal" $A_0(3)$ model.

premia. Following Duffee (2002), we allow risk premia to be affine in the state variables. This specification nests the constant risk-premium assumption made in previous empirical analysis of commodity futures (e.g., Schwartz (1997)). Existing theoretical models of commodity prices based on the theory of storage (such as RSS (2000)) assume risk neutrality, and thus make no prediction about risk premia. However, allowing for time-varying risk premia is important since, as argued by Fama and French (1987, 1988), negative correlation between risk premia and spot prices may generate mean reversion in spot prices. In the context of our affine model, allowing for risk premia to be level dependent implies that state variables have different strength of mean reversion under the historical and risk-neutral measures. Mean-reversion under the risk-neutral measure is due to convenience yields, whereas mean reversion under the historical measure results from both the convenience yield and the time variation in risk premia. The former is important to capture the cross section of futures prices, whereas the latter affects the time-series properties of spot and futures prices.

We use weekly data on crude oil, copper, gold, and silver futures contracts and U.S. treasury bills, from January 2, 1990 to August 25, 2003. We estimate the model using maximum likelihood, since it takes full advantage of the Gaussian-affine structure of our model.[4] Using standard pricing results on Gaussian-affine models (Langetieg (1980), Duffie, Pan, and Singleton (DPS) (2000)) we obtain closed-form solutions for futures, zero-coupon bond prices, and the transition density of the state vector. Results indicate that the maximal convenience yield model improves over all (nested) specifications previously investigated. Three factors are needed to capture the dynamics of futures prices. Allowing convenience yields and risk premia to be a function of the level of spot commodity prices as well as interest rates is an important feature of the data. For crude oil and copper, we find convenience yields are significantly increasing in spot commodity prices, which is in line with predictions of the theory of storage. For silver this dependence is much lower and for gold it is negligible, and for these two metals, the level of convenience yield is much lower and not particularly variable. For all commodities, the sign of the dependence of convenience yields on interest rates is positive and significant, and we find economically significant negative correlation between risk premia and spot prices. The point estimates further suggest that the contribution of time variation in risk premia to the total mean reversion strength under the historical measure is increasing in the degree to which an asset may serve as a store of value. Related, the level of convenience yields is increasing in the degree to which an asset serves for production purposes (high for oil and copper and low for gold and silver).

These results are robust to the inclusion of jumps in the spot dynamics. Specifically, we decompose the jump component of spot commodity prices into three parts. We find evidence for a high-intensity jump with stochastic jump size of approximately zero mean, and two lower-intensity jumps with constant jump

[4] The same approach has been widely used in the literature: Chen and Scott (1993), Pearson and Sun (1994), and Duffie and Singleton (1997).

sizes. The estimates of the risk-neutral drift parameters of the state vector are almost unchanged. Including jumps mainly affects the estimates of the volatility coefficients and the risk-premia parameters. Indeed, we show that jumps in the spot price have little impact on the predicted cross section of futures prices.[5] However, accounting for jumps helps to better capture the historical dynamics of futures prices.

Bessembinder et al. (1995) also find evidence for mean reversion in commodity prices by comparing the sensitivity of long-maturity futures prices to changes in spot prices (or, effectively, short-maturity futures prices). Since their test uses only information from the cross section of futures prices, it cannot detect mean reversion resulting from "movements in the risk-premium component" (see their discussion p. 362). Consequently, their test cannot determine whether historical time series of commodity prices actually exhibit mean reversion.[6] In contrast to their paper, our model allows one to disentangle the various sources of mean reversion, namely, level dependence in convenience yield vs. time variation in risk premia. Fama and French (1988) study the importance of time variation in risk premia for mean reversion in commodity prices using simple univariate linear regressions of changes in spot prices and the forward premium on the basis (similar to Fama (1984)). Their results are inconclusive for most commodities (and in particular for the metals studied here), mainly, they argue, because the basis exhibits too little volatility for regressions to reliably identify time variation in risk premia. In contrast, viewing commodity futures through the "filter" of affine models potentially allows us to obtain more reliable estimates of time variation in risk premia.[7]

Finally, we document the economic importance of disentangling the two sources of mean reversion by studying two applications, option pricing, and, value-at-risk (VAR) computations. Ignoring spot price dependence of convenience yields results in a misspecification of the risk-neutral dynamics of the spot price and can result in gross misvaluation of options. Mean-reversion under the risk-neutral measure effectively reduces the term volatility of the spot price, which tends to reduce option values. This is especially true for oil and copper, where an important fraction of the total mean reversion is due to the positive relation between spot prices and convenience yields. Comparing option prices using our parameter estimates with those obtained using a restricted model (with parameters estimated imposing the condition that convenience yields be

[5] As we show in Appendix F, jumps impact futures prices only when the convenience yield depends on the spot price. The intuition is that futures prices are martingales under the risk-neutral measure. Combined with the martingale restriction on the drift of the spot price process, this implies that jumps in the spot price can only "matter" if there is a common jump in the convenience yield (or the interest rate). Hilliard and Reis (1998), for example, find that in their model, jumps have no impact on futures prices. Their convenience yield model is not maximal however.

[6] Indeed, the risk premia could, in principle, be time varying in a way to offset the "risk-neutral" mean reversion induced by convenience yields.

[7] Piazzesi (2003) offers further discussion of the advantages of the affine framework, which explicitly imposes a cross-sectional no-arbitrage restriction, over an unrestricted vector-autoregression, for example.

linearly independent of the spot price) results in sizable errors of about 30% for at-the-money options. An implication is that for crude oil and copper investments the "naive" model will predict much higher real-option values and will tend to defer investment more than the more realistic "maximal" model.

Similarly, ignoring time variation in risk premia may lead to severe overestimation of the value-at-risk of commodity-related investments. Comparing the value-at-risk of an investment in one unit of the asset obtained when estimating the naive model versus the maximal model, we find that the tails of the distribution of the naive model tend to be fatter the longer the maturity of the investment considered. For gold and silver, we find that for a 5-year horizon investment, the loss implied by a 5% value-at-risk more than doubles when computed with models which ignore time variation in risk premia. These examples illustrate that disentangling the sources of mean reversion in risk premia can have a substantial impact on valuation, investment decisions,[8] and risk management.

The rest of the article is structured as follows. Section I presents the model. Section II discusses the specification of risk premia. Section III describes the empirical analysis and discusses the results. Section IV shows the economic implications of the model and Section V concludes.

## I. The Maximal Convenience Yield Model

In this section, we develop a general three-factor Gaussian model of (log) futures prices. Following Duffie and Kan (DK; 1996), DPS (2000), and DS (2000), we first introduce a canonical representation of a three-factor Gaussian state vector driving futures prices.[9] We assume that the spot commodity price $S(t)$ is defined by

$$X(t) := \log S(t) = \phi_0 + \phi_Y^\top Y(t), \tag{1}$$

where $\phi_0$ is a constant, $\phi_Y$ is a $3 \times 1$ vector, and $Y^\top(t) = (Y_1(t), Y_2(t), Y_3(t))$ is a vector of state variables that follows a Gaussian diffusion process under the risk-neutral measure $Q$:[10]

$$dY(t) = -\kappa^Q Y(t)\,dt + dZ^Q(t), \tag{2}$$

[8] The impact of various assumptions about the dynamics of the convenience yield on real-option valuation and investment decisions is discussed in the last section of Schwartz (1997).

[9] The model is in the $A_0(3)$ family using the terminology of DS (2000). They show that an $N$-factor affine model can be classified into $N+1$ families of models denoted $A_M(N)$ depending on the number, $M$, of state variables entering the conditional variance-covariance structure of the state vector. In Gaussian models, the conditional covariance structure is constant, $M = 0$.

[10] We assume the existence of such a risk-neutral measure. See Duffie (1996) for conditions under which the existence of such a measure is equivalent to the absence of arbitrage. If a sufficient number of futures contracts are traded, then in general, with such an affine structure, markets are complete and this martingale measure is unique. However, Collin-Dufresne and Goldstein (2002) build finite-dimensional affine models with a continuum of traded derivatives that yield incomplete markets.

where $\kappa^Q$ is a $3 \times 3$ lower triangular matrix that reflects the degree of mean reversion of the processes, and $dZ^Q$ is a $3 \times 1$ vector of independent Brownian motions. It is well known (for example, Duffie (1996)) that the futures price $F^T(t)$ at time $t$ for the purchase of one unit of commodity $S(T)$ at time $T$ is simply the expected future spot price under the risk-neutral measure. Using standard results on pricing within the affine framework (e.g., Langetieg (1980), DK (1996), DPS (2000)), we obtain the following expression

$$F^T(t) = E_t^Q\left[e^{X(T)}\right] = e^{A_F(T-t)+B_F(T-t)^\top Y(t)}, \tag{3}$$

where $A_F(\tau)$ and $B_F(\tau)$ are the solutions to the following system of ordinary differential equations:

$$\frac{dA_F(\tau)}{d\tau} = \frac{1}{2}B_F(\tau)^\top B_F(\tau)$$

$$\frac{dB_F(\tau)}{d\tau} = -\kappa^{Q\top} B_F(\tau)$$

with boundary conditions $A_F(0) = \phi_0$ and $B_F(0) = \phi_Y$ which can be solved in closed form (see Appendix A).

Such a model is maximal in the sense that, conditional on observing only futures prices (and not the state variables $Y_1, Y_2$, and $Y_3$ themselves), it has the maximum number of identifiable parameters. This result follows directly from the analysis in DS (2000). However, unlike in DS (2000) where bonds are derivatives of the nontraded short rate, in our framework, the underlying process $S(t)$ is a traded commodity. We emphasize that the assumption that we observe all futures prices implies that the spot price, which is but one particular futures price, is observable.[11] Absence of arbitrage therefore implies that

$$E_t^Q[dS(t)] = (r(t) - \delta(t))S(t)\,dt, \tag{4}$$

where $r(t)$ is the instantaneous risk-free rate and $\delta(t)$ is the instantaneous convenience yield. The latter has the standard interpretation of a dividend flow, net of storage costs, which accrues to the holder of the commodity in return for immediate ownership (e.g., Hull (2000)). As discussed in the introduction, convenience yields also arise endogenously in models based on the theory of storage (e.g., RSS (2000)) as a result of the interaction among supply, demand, and storage decisions. Augmenting the data set with bond prices and making an identifying assumption about the short-rate model that drives the term structure of interest rates, we can recover the process for the convenience yield from equation (4), and effectively view the latter as defining the convenience yield.[12]

[11] This is similar to the special role played by the short rate for identification of parameters in affine term structure models (e.g., Collin-Dufresne, Goldstein, and Jones (2002)).

[12] If the spot price is actually not a traded asset (as would be the case for electricity futures, for example), then the process $\delta$ defined by equation (4) is still of interest, as it reflects, per definition, how much the spot price dynamics differ from that of a traded asset.

Following previous empirical papers on commodity futures, we assume that the risk-free rate follows a one-factor Gaussian process:[13]

$$r(t) = \psi_0 + \psi_1 Y_1(t). \tag{5}$$

Zero-coupon bond prices may be computed explicitly by solving for $P^T(t) = \mathrm{E}_t^Q[e^{-\int_t^T r(s)\,ds}]$ as in Vasicek (1977) (see Appendix B).

Using the definitions for $X(t)$ and $r(t)$ given in equations (1) and (5), noting the arbitrage restriction (4), and applying Itô's lemma, we obtain the following expression for the maximal convenience yield implied by our model:

$$\delta(t) = r(t) - \frac{E_t^Q[dX(t)] + \frac{1}{2}V_t^Q[dX(t)]}{dt} = \psi_0 - \frac{1}{2}\phi_Y^\top \phi_Y + \psi_1 Y_1(t) + \phi_Y^\top \kappa^Q Y(t). \tag{6}$$

Equations for $X(t)$, $r(t)$, and $\delta(t)$ given in (1), (5), and (6) above specify a unique transformation from the latent variables $\{Y_1, Y_2, Y_3\}$ to $\{r, \delta, X\}$, and as such, we may derive the dynamics of the convenience yield implied by the model. We summarize the results in the following proposition:

PROPOSITION 1: *Assume the risk-free interest rate follows an autonomous one-factor Ornstein–Uhlenbeck process as in equation* (5)*. Then, the "maximal" model of futures prices and convenience yields defined as in equations* (1)–(6) *can equivalently be represented by*

$$dr(t) = \kappa_r^Q\left(\theta_r^Q - r(t)\right)dt + \sigma_r\, dZ_r^Q(t), \tag{7}$$

$$d\delta(t) = \left(\kappa_{\delta 0}^Q + \boxed{\kappa_{\delta r}^Q}\, r(t) + \kappa_\delta^Q \delta(t) + \boxed{\kappa_{\delta X}^Q}\, X(t)\right)dt + \sigma_\delta\, dZ_\delta^Q(t), \tag{8}$$

$$dX(t) = \left(r(t) - \delta(t) - \frac{1}{2}\sigma_X^2\right)dt + \sigma_X\, dZ_X^Q(t), \tag{9}$$

*where* $Z_X^Q, Z_\delta^Q$, and $Z_r^Q$ *are standard correlated Brownian motions.*

*Proof:* The proof follows immediately from applying Itô's Lemma to $X(t)$, $r(t)$, and $\delta(t)$ defined in equations (1), (5), and (6) above, and noting that these equations specify a unique transformation from the latent variables $\{Y_1, Y_2, Y_3\}$ to $\{r, \delta, X\}$. In Appendix C, we provide the relation between the parameters of the latent model and the parameters of the $\{r, \delta, X\}$ representation. For future reference, we define the correlation coefficients as follows:

$$dZ_X^Q(t)\,dZ_\delta^Q(t) = \rho_{\delta X}\,dt, \quad dZ_X^Q(t)\,dZ_r^Q(t) = \rho_{rX}\,dt, \quad \text{and}$$
$$dZ_\delta^Q(t)\,dZ_r^Q(t) = \rho_{r\delta}\,dt. \tag{10}$$

Q.E.D.

[13] This model is maximal in the $A_0(1)$ family, that is, conditional on observing only bond prices, it has the maximum number of parameters identifiable for a one-factor Gaussian model.

In the class of three-factor Gaussian models of futures (and spot) commodity prices in which the short rate is driven by one factor, this is the most general specification of the convenience yield that is also identifiable.[14]

Proposition 1 shows that, in general, the drift of the convenience yield process may depend on both the interest rate and the spot rate. This contrasts with the specifications analyzed in the existing literature which, typically, assume that the convenience yield follows an autonomous process, that is, that the highlighted coefficients in equation (8) are 0. The following proposition provides a better understanding for the significance of imposing restrictions on the parameters $\kappa_{\delta r}^{Q}$ and $\kappa_{\delta X}^{Q}$.

PROPOSITION 2: *The maximal convenience yield of Proposition* 2 *can be decomposed as*

$$\delta(t) = \hat{\delta}(t) + \alpha_r r(t) + \alpha_X X(t), \tag{11}$$

*where* $\hat{\delta}$ *follows an autonomous Ornstein–Uhlenbeck process:*

$$d\,\hat{\delta}(t) = \kappa_{\hat{\delta}}^{Q}\left(\theta_{\hat{\delta}}^{Q} - \hat{\delta}(t)\right)dt + \sigma_{\hat{\delta}}\,dZ_{\hat{\delta}}^{Q}(t). \tag{12}$$

*There is such a unique decomposition, where*

$$\begin{cases} \alpha_r = 0 \iff \kappa_{\delta r}^{Q} = 0, \\ \alpha_X = 0 \iff \kappa_{\delta X}^{Q} = 0. \end{cases}$$

*Using the above decomposition, the dynamics of the spot price process become*

$$dX(t) = \left(\alpha_X\left(\theta_X^{Q} - X(t)\right) + (\alpha_r - 1)\left(\theta_r^{Q} - r(t)\right) + \theta_{\hat{\delta}}^{Q} - \hat{\delta}(t)\right)dt + \sigma_X\,dZ_X^{Q}(t), \tag{13}$$

*where the long-term mean of the log spot price is given by* $\theta_X^{Q} = \frac{1}{\alpha_X}((1-\alpha_r)\theta_r^{Q} - \theta_{\hat{\delta}}^{Q} - \frac{1}{2}\sigma_X^2)$.

*Proof:* Applying Itô's lemma to the right-hand side of equation (11) and equating drift and diffusion of the resulting process with those of equation (8) shows that there exist two possible proposed decompositions, which are given by

$$\alpha_X^{\pm} = \frac{1}{2}\left(-\kappa_{\hat{\delta}}^{Q} \pm \sqrt{\left(\kappa_{\hat{\delta}}^{Q}\right)^2 - 4\kappa_{\delta X}^{Q}}\right), \tag{14}$$

$$\alpha_r^{\pm} = \frac{\alpha_X^{\pm} - \kappa_{\delta r}^{Q}}{\alpha_X^{\pm} + \kappa_r^{Q} + \kappa_{\hat{\delta}}^{Q}}, \tag{15}$$

[14] Note that unlike in DS (2000), we have a three-state variable model of two types of securities, namely, bond and futures prices. Even though the two models are separately maximal, one may wonder if together they form a maximal model, as the joint observation of the two securities may allow the empiricist to recover more information about the state vector than observing the two separately. It turns out that in the Gaussian case, the joint observation of two types of securities does not help identify more parameters. The model above is thus maximal, conditional on observing bond and futures prices and restricting the term structure to be driven by only one factor.

$$\kappa_{\hat{\delta}}^{Q\pm} = -\kappa_{\delta}^{Q} - \alpha_X^{\pm}, \tag{16}$$

$$\kappa_{\hat{\delta}}^{Q\pm}\theta_{\hat{\delta}}^{Q\pm} = \kappa_{\delta 0}^{Q} + \alpha_X^{\pm}\sigma_X^2/2 - \alpha_r^{\pm}\kappa_r^{Q}\theta_r^{Q}, \tag{17}$$

$$\sigma_{\hat{\delta}}^{\pm}\, dZ_{\hat{\delta}}^{Q\pm} = \sigma_\delta\, dZ_\delta^{Q} - \alpha_X^{\pm}\sigma_X\, dZ_X^{Q} - \alpha_r^{\pm}\sigma_r\, dZ_r^{Q}, \tag{18}$$

$$\left(\sigma_{\hat{\delta}}^{\pm}\right)^2 = \sigma_\delta^2 + \left(\alpha_X^{\pm}\right)^2\sigma_X^2 + \left(\alpha_r^{\pm}\right)^2\sigma_r^2 \tag{19}$$

$$- 2\rho_{\delta X}\sigma_\delta\sigma_X\alpha_X^{\pm} + 2\rho_{rX}\alpha_r^{\pm}\alpha_X^{\pm}\sigma_r\sigma_X - 2\rho_{r\delta}\sigma_\delta\alpha_r^{\pm}\sigma_r. \tag{20}$$

Defining $\varsigma = \text{sign}(\kappa_\delta^Q)$, we see that only the solution $\alpha_X^{\varsigma}, \alpha_r^{\varsigma}, \kappa_{\hat{\delta}}^{Q\,\varsigma}$, and $\theta_{\hat{\delta}}^{Q^\varsigma}$ satisfies the conditions $\alpha_X = 0 \iff \kappa_{\delta X}^{Q} = 0$ and $\alpha_r = 0 \iff \kappa_{\delta r}^{Q} = 0$. Further, we note that $\alpha_r, \alpha_X$ are real if and only if $\kappa_\delta^{Q^2} - 4\kappa_{\delta X}^{Q} \geq 0$, which corresponds to the condition that eigenvalues of the mean reversion matrix must be real.

Finally, we note that $Z_{\hat{\delta}}^{Q}$ as defined by equations (18) and (20) is a standard Brownian motion which is correlated with $Z_X^Q, Z_r^Q$. For future reference, we define the correlation coefficients, as

$$dZ_X^Q(t)\, dZ_{\hat{\delta}}^{Q}(t) = \rho_{\hat{\delta}X}\, dt, \quad dZ_{\hat{\delta}}^{Q}(t)\, dZ_r^Q(t) = \rho_{r\hat{\delta}}\, dt. \tag{21}$$

Q.E.D.

The two propositions above show that once we assume that the short rate follows an autonomous one-factor process, then the arbitrage restriction (4) delivers a convenience yield process that has its own specific stochastic component $\hat{\delta}$, but, which is also linearly affected by the short rate and the log spot price.[15] Proposition 2 also makes apparent that the maximal model nests the three models analyzed in Schwartz (1997), as well as the models of Ross (1997), Brennan and Schwartz (1985), Gibson and Schwartz (1990), and Schwartz and Smith (2000).[16] For example, Schwartz's model 1 corresponds to a one-factor ($X$) model with $\alpha_r = 0$. Schwartz's model 2 corresponds to a two-factor model $(X, \hat{\delta})$ with $\alpha_r = \alpha_X = 0$. Schwartz's model 3 corresponds to a three-factor model with $\alpha_r = \alpha_X = 0$.

One simple insight of the maximal convenience yield model is that the one-factor models of Ross (1997) and Schwartz (1997), which allow for mean reversion under the risk-neutral measure of spot prices, can be interpreted as arbitrage-free models of commodity spot prices, where the convenience yield is a function of the log spot price. A positive relation between the convenience yield and the (log) spot price, that is, a positive $\alpha_X$, leads to a mean-reverting spot

[15] Note that it seems economically sensible to assume that a market-wide variable such as the short rate follows an autonomous process, that is, is not driven by the convenience yield or the spot price of a specific commodity. The model could easily be extended to allow for multifactor term structure models. However, to maintain the assumption that interest rate risk is autonomous and at the same time have the convenience yield and spot price be specific sources of risk would require a four-factor model.

[16] Some of these are actually nested in the models analyzed by Schwartz (1997).

price under the risk-neutral measure. The latter feature seems to be empirically desirable to fit the cross section of futures prices. A positive relation between convenience yield and spot price also seems to be consistent with the predictions of theoretical models. Several papers (Working (1949), Brennan (1958), Deaton and Laroque (1992), and RSS (2000)) have shown that convenience yields arise endogenously as a result of the interaction among supply, demand, and storage decisions. In particular, RSS (2000) show that, in a competitive rational expectations model of storage, when storage in the economy is driven to its lower bound, for example, in periods of relative scarcity of the commodity available for trading, convenience yields should be high. This provides some economic rationale for allowing the convenience yield to depend on spot prices as in our maximal model. In fact, assuming that periods of low inventory and relative scarcity of the commodity coincide with high spot prices, the theory of storage predicts a positive relation between convenience yields and spot prices.

Further, RSS (2000) note that the correlation structure between spot prices and convenience yields should be time varying, in contrast to the prediction of standard commodity derivatives pricing models such as Brennan (1991), Gibson and Schwartz (1990), Amin et al. (1995), and Schwartz (1997). Since it is a Gaussian model, the maximal convenience yield has a constant instantaneous correlation structure. However, since all state variables enter the drift of convenience yield and spot price, it allows for a richer unconditional correlation structure than previous specifications.[17]

Finally, note that previous models restrict $\alpha_r$ to be 0, that is, restrict convenience yields to be independent of the level of interest rates. While most theoretical models assume zero interest rates (e.g., RSS (2000)) and thus do not deliver empirical predictions about that coefficient, relaxing this assumption seems desirable. If we expect interest rates and inventory to be correlated, then, following the theory of storage argument, we may expect a significant nonzero coefficient. In fact, to the extent that holding inventory becomes more costly in periods of high interest rates, we may expect a negative correlation between interest rates and inventory, and thus a positive $\alpha_r$.

Of course, our model is a reduced-form model which makes no predictions about these relations. However, it is the natural framework to empirically investigate these questions. In the next sections, we discuss the specification of risk premia and empirical implementation.

## II. Specification of Risk Premia

Our discussion above is entirely cast in terms of the risk-neutral dynamics of state variables. These are useful to price the cross section of futures prices. To explain the historical time-series dynamics of prices and to subject our model

[17] In RSS (2000), the correlation is derived endogenously and is a function of the level of inventory. To the extent that spot prices proxy for inventories, the maximal convenience yield model may be able to capture that feature. It is a reduced form model, however, and inventory is not an explicit state variable of the model.

to empirical scrutiny, we need a specification of risk premia. In contrast to previous empirical research (e.g., Schwartz (1997)) which assumes constant risk premia, we allow risk premia to be a linear function of the state variables, following Duffee (2002) and DS (2002).

For the canonical representation, we choose the following specification for the risk premia:

$$dZ^Q(t) = dZ(t) + (\beta_{0Y} + \beta_{1Y} Y(t))\,dt. \tag{22}$$

Here, $Z$ is a $3 \times 1$ vector of Brownian motions on a standard filtered probability space $(\Omega, \mathcal{F}, P)$, $\beta_{0Y}$ is a $3 \times 1$ vector of constants, and $\beta_{1Y}$ is a $3 \times 3$ matrix of constants. The process under the physical $P$-measure is given by

$$dY(t) = (\beta_{0Y} - \left(\kappa^Q - \beta_{1Y})Y(t)\right)dt + dZ(t). \tag{23}$$

It is well known (e.g., Liptser and Shiryaev (1977) Theorem 7.15, p. 279) that in this Gaussian framework, the transformation in (22) above defines a measure $Q$ that is equivalent to the physical measure $P$ and under which $Z^Q$ is a vector of standard Brownian motions. In other words, the $Q$-measure is an "equivalent risk-neutral measure" (EMM) for the economy described by the $P$-measure dynamics above. The existence of an EMM is sufficient to rule out arbitrage opportunities (Harrison and Kreps (1979)).

With this specification the dynamics of the state variables are Gaussian under both the historical and risk-neutral measure. Note that both the mean reversion coefficient and the long-run mean differ under both measures. Under the physical measure, the mean reversion matrix is $(\kappa^Q - \beta_{1Y})$ and the long-run mean vector is $(\kappa^Q - \beta_{1Y})^{-1}\beta_{0Y}$. Traditional models assume that $\beta_{1Y} = 0$.[18]

For ease of economic interpretation, we prefer to study the $\{r, \hat{\delta}, X\}$ representation obtained in Proposition 2.[19] The risk-premium specification of equation (22) can equivalently be rewritten in terms of the rotated Brownian motion basis (see Appendix D) as

$$d\begin{pmatrix} Z_r^Q \\ Z_{\hat{\delta}}^Q \\ Z_X^Q \end{pmatrix} = d\begin{pmatrix} Z_r \\ Z_{\hat{\delta}} \\ Z_X \end{pmatrix} + \Sigma^{-1}\left(\begin{pmatrix} \beta_{0r} \\ \beta_{0\hat{\delta}} \\ \beta_{0X} \end{pmatrix} + \begin{pmatrix} \beta_{rr} & \beta_{r\hat{\delta}} & \beta_{rX} \\ \beta_{\hat{\delta}r} & \beta_{\hat{\delta}\hat{\delta}} & \beta_{\hat{\delta}X} \\ \beta_{Xr} & \beta_{X\hat{\delta}} & \beta_{XX} \end{pmatrix}\begin{pmatrix} r(t) \\ \hat{\delta}(t) \\ X(t) \end{pmatrix}\right)dt, \tag{24}$$

[18] Duffee (2002) shows that the more general, "essentially affine" specification, improves the ability of term structure models at capturing the predictability of bond price returns under the historical measure, while retaining their ability at pricing the cross section of bonds (i.e., fitting the shape of the term structure).

[19] As is apparent from the proof of Proposition 2, studying this particular decomposition of the "maximal" convenience yield model of Proposition 1 effectively restricts the model to the parameter set for which the eigenvalues of the mean reversion matrix are real. We check empirically (by directly estimating the model of Proposition 1) that this restriction was never binding for our data.

where

$$\Sigma = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_{\hat{\delta}} & 0 \\ 0 & 0 & \sigma_X \end{pmatrix}. \tag{25}$$

Given the representation adopted, it seems natural to impose the following restrictions on the risk premia:

$$\begin{cases} \beta_{r\hat{\delta}} = \beta_{rX} = 0, \\ \beta_{\hat{\delta}r} = \beta_{\hat{\delta}X} = 0. \end{cases} \tag{26}$$

The first set of restrictions basically guarantees that the risk-free interest rate's term premia do not depend on the level of convenience yield or commodity spot price. This ensures that the short rate follows an autonomous process under both measures. It is also consistent with applying this model to different commodities which all share the same interest rate model. The second set of restrictions simply guarantees that the component of the convenience yield ($\hat{\delta}$) that is linearly independent of interest rate and spot price level under the risk-neutral measure, remains so under the historical measure. With these restrictions, the dynamics of the state variables $\{r, \hat{\delta}, X\}$ under the historical measure have the same form as under the risk-neutral measure, but with different risk-adjusted drift coefficients.

PROPOSITION 3: *If risk premia are given by equations* (24)–(26)*, then the state variables* $\{r, \hat{\delta}, X\}$ *introduced in Proposition* 2 *have the following dynamics under the physical measure:*

$$dr(t) = \kappa_r^P\left(\theta_r^P - r(t)\right)dt + \sigma_r\, dZ_r(t), \tag{27}$$

$$d\hat{\delta}(t) = \kappa_{\hat{\delta}}^P\left(\theta_{\hat{\delta}}^P - \hat{\delta}(t)\right)dt + \sigma_{\hat{\delta}}\, dZ_{\hat{\delta}}(t), \tag{28}$$

$$dX(t) = \left(\mu(t) - \delta(t) - \frac{1}{2}\sigma_X^2\right)dt + \sigma_X\, dZ_X(t), \tag{29}$$

$$:= \left(\kappa_{Xr}^P\left(\theta_r^P - r(t)\right) + \kappa_{X\hat{\delta}}^P\left(\theta_{\hat{\delta}}^P - \hat{\delta}(t)\right) + \kappa_X^P\left(\theta_X^P - X(t)\right)\right)dt + \sigma_X\, dZ_X(t), \tag{30}$$

*where* $\delta$ *is defined in equation* (11)*, and* $Z_X$*,* $Z_{\hat{\delta}}$*, and* $Z_r$ *are standard Brownian motions defined in equation* (24)*.*

*The relation between P and Q parameters expressed in terms of the risk premia is*

$$\kappa_r^P = \kappa_r^Q - \beta_{rr}, \quad \theta_r^P = \frac{\kappa_r^Q\theta_r^Q + \beta_{0r}}{\kappa_r^Q - \beta_{rr}}, \tag{31}$$

$$\kappa_{\hat{\delta}}^P = \kappa_{\hat{\delta}}^Q - \beta_{\hat{\delta}\hat{\delta}}, \quad \theta_{\hat{\delta}}^P = \frac{\kappa_{\hat{\delta}}^Q\theta_{\hat{\delta}}^Q + \beta_{0\hat{\delta}}}{\kappa_{\hat{\delta}}^Q - \beta_{\hat{\delta}\hat{\delta}}}, \tag{32}$$

$$\mu(t) = r(t) + (\beta_{0X} + \beta_{Xr}r(t) + \beta_{X\hat{\delta}}\hat{\delta}(t) + \beta_{XX}X(t)), \tag{33}$$

$$\kappa_{Xr}^{P} = \kappa_{Xr}^{Q} - \beta_{Xr}, \quad \kappa_{X\hat{\delta}}^{P} = 1 - \beta_{X\hat{\delta}}, \quad \kappa_{X}^{P} = \alpha_{X} - \beta_{XX}, \tag{34}$$

$$\theta_{X}^{P} = \frac{\theta_{X}^{Q}\alpha_{X} + \beta_{0X}}{\alpha_{X} - \beta_{XX}} + \frac{\kappa_{Xr}^{Q}\theta_{r}^{Q} - \kappa_{Xr}^{P}\theta_{r}^{P} + \theta_{\hat{\delta}}^{Q} - \kappa_{X\hat{\delta}}^{P}\theta_{\hat{\delta}}^{P}}{\alpha_{X} - \beta_{XX}}. \tag{35}$$

Proposition 3 above shows that allowing for essentially affine risk premia allows one to disentangle the level of mean reversion in spot commodity prices under the risk-neutral measure from the level of mean reversion under the historical measure. The former is essential to capture the term structure of futures prices (i.e., the cross section), whereas the latter captures the time-series properties of spot commodity prices. Fama and French (1987, 1988) argue that negative correlation between risk premia and spot prices can generate mean reversion in spot prices. Our model captures this feature as is apparent from equations (29) and (33). A negative $\beta_{XX}$ implies negative correlation between risk premia and spot prices and generates mean reversion in spot prices. Thus, our model has the ability to distinguish two sources of mean reversion. First, mean reversion in (log) spot prices can be due to level dependence in convenience yield (a positive $\alpha_X$). Second, mean reversion can appear as a result of negative correlation between risk premia and spot prices (a negative $\beta_{XX}$). Only the convenience yield component affects the cross section of futures prices, that is, enters the risk-neutral measure dynamics. Both drive the time series of commodity prices, that is, enter the historical measure price dynamics. In addition, the instantaneous correlation of the spot price with the risk-free interest rate and $\hat{\delta}$, combined with the signs of, respectively, $\kappa_{Xr}$ and $\kappa_{X\hat{\delta}}$, may contribute to "mean-reversion-like" behavior in commodity prices. Distinguishing between the various sources (if any) of mean reversion may have important consequences for valuation and investment decisions, as well as risk management, as we document below. However, we first turn to the empirical estimation of the model.

## III. Empirical Implementation

We estimate our model for four types of commodity futures using maximum likelihood. We first describe the data, then the empirical methodology. We then discuss our results.

### *A. Description of the Data*

Our data set consists of futures contracts on crude oil, copper, gold, silver, and zero-coupon bond prices.[20] For all commodities, we use weekly data from

[20] The data for the commodities are from the New York Mercantile Exchange. The crude oil data are from the NYMEX Division, while the copper, gold, and silver data are from the COMEX Division. The interest rate data are from the Federal Reserve Board.

January 2, 1990 to August 25, 2003.[21] Table I contains the summary statistics for the four commodities. The maturities of the contracts studied differ across commodity. We use short-term contracts with maturities 1, 3, 6, 9, 12, 15, and 18 months (labeled from F01 to F18), and depending on availability, we also include longer-maturity contracts. For crude oil, copper, gold, and silver, we use long-term contracts with maturities up to 36, 24, 48, and 48 months, respectively. These long-term data are not fully available for the whole period studied (713 weeks) since many of these contracts were not available in 1990. If a specific contract is missing, we select the one with the nearest maturity. A special characteristic of futures contracts is that the last trading day is a specific day of each month, implying that the maturity of the contracts varies over time.[22] For interest rates we use constant-maturity Treasury yields to build zero-coupon bonds with maturities of 0.5, 1, 2, 3, 5, 7, and 10 years.

Figure 1 shows the price of the $F01$ and $F18$ contracts for crude oil, copper, gold, and silver. We can see a decreasing tendency on copper and gold prices during the period analyzed. Also, copper, oil, and gold each reached their lowest price in the period during the first half of 1999. Finally, if we casually compare the $F01$ and $F18$ contracts, there appears to be mean reversion (under the risk-neutral measure) in copper and crude oil prices. Indeed, the difference between the $F18$ and $F01$ futures prices alternates signs.[23] Because of convergence, the $F01$ futures price should be close to the spot price. Thus, alternating signs in $F18 - F01$ suggest periods of strong backwardation in oil and copper markets as documented in Litzenberger and Rabinowitz (1995). Gold and silver exhibit fewer episodes of strong backwardation. The "basis" estimated by $F18 - F01$ appears to be more stable and mostly positive. Figure 2 plots the term structures for each commodity and confirms these findings. It seems that oil and copper have higher degrees of mean reversion (under the risk-neutral measure) than gold and silver. Figure 3 presents the historical evolution of the 6- and 60-month interest rates used for the estimation.

[21] For simplicity, we do not adjust the data for inflation. Given the relative short time span of our data set this seems reasonable. A more realistic implementation of the model with a longer time series would require explicit modeling of inflation for two reasons. First, it seems more likely to find mean reversion in real prices rather than nominal prices. Second, with our specification, the convenience *yield* is a linear function of the log spot price, which seems more sensible if prices are expressed in real terms (we thank Mark Rubinstein for pointing this out).

[22] The last trading day is different across commodities. For copper, gold, and silver the last trading day is the close of the third-last business day of the maturing delivery month, while for crude oil it is the close of the third business day prior to the 25th calendar day of the month preceding the delivery month.

[23] Suppose that $d \ln S_t = (r - \delta - \kappa \ln S_t - \frac{1}{2}\sigma^2)\, dt + \sigma\, dZ_t^Q$, where all coefficients are constant. Then, simple calculations show that $F^T(t) = E_t^Q[S_T] = \exp(\theta + (\ln S_t - \theta)e^{-\kappa(T-t)} + \frac{\sigma^2}{2} B_{2\kappa}(T - t))$, where $\kappa\theta = r - \delta - \frac{1}{2}\sigma^2$ and $B_\kappa(\tau) = (1 - e^{-\kappa\tau})/\kappa$. It is thus clear that $F18 > F01 \Leftrightarrow \ln S_t < \theta + \frac{\sigma^2(e^{-2\kappa T_1} - e^{-2\kappa T_{18}})}{4\kappa(e^{-\kappa T_1} - e^{-\kappa T_{18}})}$. Further, in the absence of mean reversion under the risk-neutral measure ($\kappa = 0$), we observe that $F18 - F01$ has the same (constant) sign as $r - \delta$.

**Table I**

## Statistics of Crude Oil, Copper, Gold, and Silver Futures Contracts

Statistics for weekly observations of crude oil, copper, gold, and silver futures contracts from January 2, 1990 to August 25, 2003. Oil prices are in dollars per barrel, copper prices are in cents per pound, gold prices are in dollars per troy ounce, silver prices are in cents per troy ounce, and maturities are in years. The contract denomination specifies the number of months to maturity (i.e., F03 is the 3 months to maturity futures contract).

| Crude Oil Data | | | | Copper Data | | | |
|---|---|---|---|---|---|---|---|
| Contract | Number of Observations | Mean Price (SE) | Maturity (SE) | Contract | Number of Observations | Mean Price (SE) | Maturity (SE) |
| F01 | 713 | 21.98 | 0.037 | F01 | 713 | 94.32 | 0.037 |
| | | (5.47) | (0.024) | | | (21.05) | (0.024) |
| F03 | 713 | 21.60 | 0.203 | F03 | 713 | 93.65 | 0.203 |
| | | (4.87) | (0.025) | | | (19.66) | (0.024) |
| F06 | 713 | 21.10 | 0.454 | F06 | 713 | 92.67 | 0.454 |
| | | (4.16) | (0.025) | | | (17.66) | (0.025) |
| F09 | 713 | 20.72 | 0.704 | F09 | 713 | 91.84 | 0.704 |
| | | (3.65) | (0.025) | | | (15.98) | (0.025) |
| F12 | 713 | 20.45 | 0.954 | F12 | 713 | 91.16 | 0.954 |
| | | (3.26) | (0.027) | | | (14.60) | (0.028) |
| F15 | 713 | 20.25 | 1.205 | F15 | 713 | 90.64 | 1.202 |
| | | (2.96) | (0.027) | | | (13.54) | (0.045) |
| F18 | 713 | 20.11 | 1.451 | F18 | 713 | 90.32 | 1.450 |
| | | (2.71) | (0.036) | | | (12.76) | (0.048) |
| F24 | 581 | 19.82 | 1.946 | F24 | 223 | 79.51 | 1.952 |
| | | (2.51) | (0.072) | | | (5.19) | (0.024) |
| F30 | 581 | 19.76 | 2.434 | | | | |
| | | (2.27) | (0.127) | | | | |
| F36 | 581 | 19.74 | 2.830 | | | | |
| | | (2.11) | (0.198) | | | | |

| Gold Data | | | | Silver Data | | | |
|---|---|---|---|---|---|---|---|
| Contract | Number of Observations | Mean Price (SE) | Maturity (SE) | Contract | Number of Observations | Mean Price (SE) | Maturity (SE) |
| F01 | 713 | 336.96 | 0.038 | F01 | 713 | 478.02 | 0.038 |
| | | (44.24) | (0.025) | | | (57.12) | (0.026) |
| F03 | 713 | 338.73 | 0.204 | F03 | 713 | 480.96 | 0.203 |
| | | (44.57) | (0.031) | | | (57.46) | (0.028) |
| F06 | 713 | 342.08 | 0.456 | F06 | 713 | 486.16 | 0.445 |
| | | (45.47) | (0.062) | | | (57.54) | (0.063) |
| F09 | 713 | 345.40 | 0.705 | F09 | 713 | 491.13 | 0.697 |
| | | (46.47) | (0.063) | | | (57.57) | (0.064) |
| F12 | 713 | 348.77 | 0.954 | F12 | 713 | 495.96 | 0.944 |
| | | (47.55) | (0.064) | | | (57.82) | (0.062) |
| F15 | 713 | 352.29 | 1.206 | F15 | 713 | 500.96 | 1.196 |
| | | (48.60) | (0.066) | | | (58.24) | (0.064) |
| F18 | 713 | 355.91 | 1.454 | F18 | 713 | 506.15 | 1.444 |
| | | (49.84) | (0.068) | | | (58.70) | (0.069) |
| F24 | 713 | 362.57 | 1.895 | F24 | 713 | 515.00 | 1.877 |
| | | (51.85) | (0.110) | | | (59.96) | (0.109) |
| F30 | 666 | 365.50 | 2.427 | F30 | 565 | 532.06 | 2.433 |
| | | (51.89) | (0.152) | | | (60.41) | (0.174) |
| F36 | 666 | 373.25 | 2.917 | F36 | 565 | 541.06 | 2.919 |
| | | (54.23) | (0.157) | | | (65.19) | (0.196) |
| F48 | 666 | 389.43 | 3.901 | F48 | 565 | 561.50 | 3.908 |
| | | (59.06) | (0.161) | | | (77.37) | (0.188) |

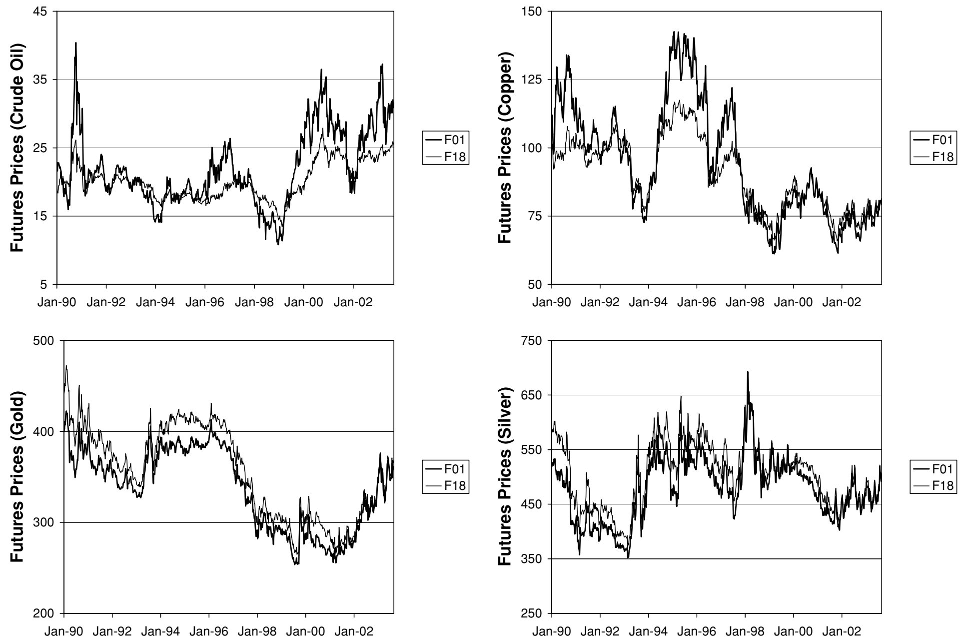


**Figure 1. Futures prices.** One-month ($F01$) and 18-month ($F18$) futures contracts on crude oil, copper, gold, and silver from January 2, 1990 to August 25, 2003. Oil prices are in dollars per barrel, copper prices are in cents per pound, gold prices are in dollars per troy ounce, and silver prices are in cents per troy ounce.

## *B. Empirical Methodology*

We use maximum-likelihood estimation using both time-series and cross sectional data in the spirit of Chen and Scott (1993) and Pearson and Sun (1994).[24] Since the three state variables $\{r, \hat{\delta}, X\}$ are not directly observed in our data set, their approach consists in arbitrarily choosing three securities to pin down the state variables. Instead, we follow Collin-Dufresne et al. (2002) and choose to fit the first principal component of the term structure of interest rates and the first two principal components of the futures curve. Since the principal components remain affine in the state variables, they can easily be inverted for the state variables using the closed-form formulas given in Appendices A and B, which depend on the risk-neutral parameters. The remaining principal components of the term structure and of futures prices, which, at any point in time, are also deterministic functions of the state variables, are then overidentified.[25]

[24] Duffie and Singleton (1997), Collin-Dufresne and Solnik (2001), and Duffee (2002) use a similar method. Schwartz (1997) uses a Kalman-filter approach.

[25] The principal components can be thought of as portfolios of contracts with different maturities. The first principal component is in general an equally weighted portfolio of contracts, while the second principal component is a portfolio with weights that are linearly decreasing with maturity. See Collin-Dufresne et al. (2002) for further details on the procedure.

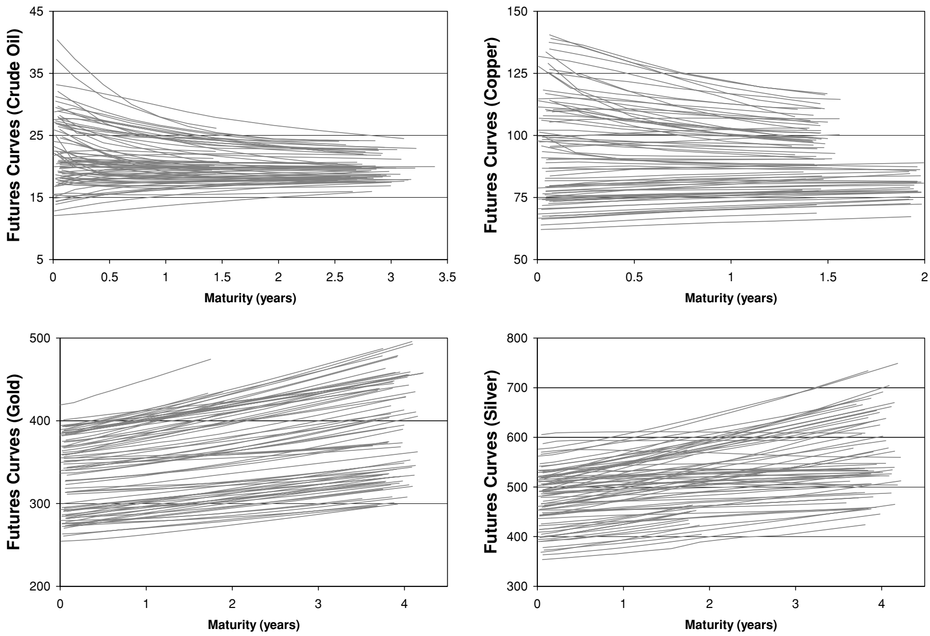


**Figure 2. Futures curves.** Monthly term structures of futures prices on crude oil, copper, gold, and silver from January 2, 1990 to August 25, 2003. Oil prices are in dollars per barrel, copper prices are in cents per pound, gold prices are in dollars per troy ounce, and silver prices are in cents per troy ounce.

Following Chen and Scott (1993), we assume these principal components are observed with "measurement errors," which we assume follow an AR(1) process. For simplicity, we assume that measurement errors in the principal components of the future prices have the same autocorrelation coefficient $\rho_F$. Similarly, we estimate only one autocorrelation coefficient for risk-free term structure errors ($\rho_P$). Given the known Gaussian transition density for the state variables and the distribution for the error terms, the likelihood can be derived.[26] We note that the transition density depends on the historical measure parameters. Apart from the likelihood value itself, the resulting properties of the measurement errors provide direct (mis-)specification tests for the model. Since long-term futures contracts are not always available, we back out the factors from the principal components of the data that are fully available for the whole period studied (713 weeks). For crude oil and copper, we use the contracts with maturity up to 18 months while for gold and silver we use the contracts with maturity up to 24 months. The remaining long-term contracts are assumed to be observed with measurement errors.

[26] Further details are provided in Appendix E.


**Figure 3. Interest rates.** Six-month and 60-month interest rates from constant maturity treasury bills from January 2, 1990 to August 25, 2003.

### *C. Empirical Results*

Table II presents the maximum-likelihood estimates of the "maximal" convenience model presented in Propositions 2 and 3. For each Commodity, we present the risk-neutral parameters that affect the drift of spot price, convenience yield, and interest rate processes under the risk-neutral measure, the risk-premia parameters, the volatility and correlation parameters, and the autocorrelation coefficients of the measurement errors of futures and Treasury rates. Table III presents the likelihood-ratio test results for three different sets of restrictions compared to the maximal model. In Table IV, we also report the point estimates of the drift parameters of the various processes under the historical measure. As shown in Proposition 3, these are simple transformations of the risk-neutral and risk-premia parameters given in Table II (e.g., $\kappa_X^P = \alpha_X - \beta_{XX}$). Finally, Table V reports point estimates for the unconditional first and second moments (long-term mean and covariances) of convenience yield and log spot prices. In the same table, we also present the long-term spot prices.[27]

Table II shows that all risk-neutral parameters are significant except for some of the correlation coefficients $\rho_{r\hat{\delta}}$, $\rho_{rX}$. This suggests that three factors are indeed necessary to explain the dynamics of each of the four commodities and, further, that innovations in the risk-free interest rates are uncorrelated with innovations in commodity spot prices and convenience yields (i.e., the

[27] Given the Gaussian nature of our model, it is straightforward to calculate the exact moments for the state variables $\{r, \hat{\delta}, X\}$. For the long-term spot price we use $E[\exp(X)] = \exp(E[X] + \frac{1}{2}\text{Var}[X])$.

**Table II**

## Maximum-Likelihood Parameter Estimates for the Maximal Model

Maximum-likelihood parameter estimates for the maximal model given in Propositions 2 and 3 in the paper, for crude oil, copper, gold, and silver weekly prices and interest rate data from January 2, 1990 to August 25, 2003.

| Parameter | Crude Oil Estimate (SE) | Copper Estimate (SE) | Gold Estimate (SE) | Silver Estimate (SE) |
|---|---|---|---|---|
| $\kappa_r^Q$ | 0.027 | 0.035 | 0.032 | 0.033 |
| | (0.007) | (0.007) | (0.007) | (0.007) |
| $\kappa_{\hat{\delta}}^Q$ | 1.191 | 1.048 | 0.392 | −0.157 |
| | (0.023) | (0.038) | (0.035) | (0.008) |
| $\alpha_r$ | 1.764 | 0.829 | 0.332 | 0.326 |
| | (0.083) | (0.097) | (0.046) | (0.101) |
| $\alpha_X$ | 0.248 | 0.150 | 0.000 | 0.085 |
| | (0.010) | (0.015) | (0.000) | (0.007) |
| $\theta_r^Q$ | 0.057 | 0.118 | 0.095 | 0.111 |
| | (0.030) | (0.015) | (0.017) | (0.016) |
| $\theta_{\hat{\delta}}^Q$ | −0.839 | −0.673 | −0.009 | −0.530 |
| | (0.033) | (0.063) | (0.003) | (0.043) |
| $\beta_{0r}$ | 0.003 | 0.002 | 0.002 | 0.000 |
| | (0.009) | (0.012) | (0.009) | (0.012) |
| $\beta_{0\hat{\delta}}$ | −1.047 | −0.435 | 0.004 | −0.510 |
| | (0.367) | (0.348) | (0.005) | (0.178) |
| $\beta_{0X}$ | 1.711 | 5.142 | 1.858 | 12.466 |
| | (0.964) | (1.956) | (1.539) | (3.381) |
| $\beta_{rr}$ | −0.137 | −0.173 | −0.140 | −0.113 |
| | (0.165) | (0.165) | (0.162) | (0.226) |
| $\beta_{\hat{\delta}\hat{\delta}}$ | −1.660 | −0.749 | −1.143 | −0.962 |
| | (0.480) | (0.531) | (0.455) | (0.322) |
| $\beta_{Xr}$ | | | −2.857 | |
| | | | (2.452) | |
| $\beta_{X\hat{\delta}}$ | | 1.919 | | 6.051 |
| | | (0.929) | | (3.733) |
| $\beta_{XX}$ | −0.498 | −0.859 | −0.301 | −1.503 |
| | (0.313) | (0.338) | (0.271) | (0.471) |
| $\sigma_r$ | 0.009 | 0.009 | 0.009 | 0.009 |
| | (0.000) | (0.000) | (0.000) | (0.000) |
| $\sigma_{\hat{\delta}}$ | 0.384 | 0.178 | 0.015 | 0.019 |
| | (0.013) | (0.006) | (0.001) | (0.001) |
| $\sigma_X$ | 0.397 | 0.228 | 0.132 | 0.223 |
| | (0.012) | (0.006) | (0.004) | (0.006) |
| $\rho_{\hat{\delta}X}$ | 0.795 | 0.588 | 0.295 | −0.422 |
| | (0.015) | (0.035) | (0.034) | (0.061) |
| $\rho_{r\hat{\delta}}$ | −0.009 | 0.107 | −0.047 | 0.019 |
| | (0.031) | (0.038) | (0.051) | (0.087) |
| $\rho_{rX}$ | 0.051 | 0.143 | −0.061 | 0.065 |
| | (0.033) | (0.037) | (0.037) | (0.044) |
| $\rho_F$ | 0.796 | 0.699 | 0.813 | 0.800 |
| | (0.011) | (0.013) | (0.011) | (0.010) |
| $\rho_P$ | 0.993 | 0.986 | 0.989 | 0.987 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| Log-likelihood | 57,153.5 | 51,761.5 | 72,899.8 | 66,114.9 |

**Table III**

## Likelihood-Ratio Tests

Likelihood ratios for the maximal model and: (a) a model in which the convenience yield is not affected by interest rates and spot prices, (b) a model with constant risk premia, and (c) both constraints together. The 5% significance level for these constraints is given by $\text{Prob}\{\chi_2^2 \geq 5.99\} = 0.05$, $\text{Prob}\{\chi_5^2 \geq 11.07\} = 0.05$, and $\text{Prob}\{\chi_7^2 \geq 14.07\} = 0.05$, respectively.

| Restriction | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| $\alpha_r = \alpha_X = 0$[a] | 1,047.20 | 312.78 | 5.60 | 66.13 |
| $\beta_{1Y} = 0$[b] | 13.16 | 12.06 | 20.01 | 21.27 |
| $\alpha_r = \alpha_X = 0$ and $\beta_{1Y} = 0$[c] | 1,057.92 | 328.02 | 23.96 | 87.81 |

**Table IV**

## Maximal Model Estimates of Historical Parameters

Maximal model estimates of historical parameters for crude oil, copper, gold, and silver using weekly prices and interest rate data from January 2, 1990 to August 25, 2003.

| Parameter | Crude Oil Estimate | Copper Estimate | Gold Estimate | Silver Estimate |
|---|---|---|---|---|
| $\kappa_r^P$ | 0.165 | 0.208 | 0.172 | 0.147 |
| $\kappa_{\hat{\delta}}^P$ | 2.850 | 1.797 | 1.535 | 0.805 |
| $\kappa_{Xr}^P$ | 0.764 | −0.171 | 2.189 | −0.674 |
| $\kappa_{X\hat{\delta}}^P$ | 1.000 | −0.919 | 1.000 | −5.051 |
| $\kappa_X^P$ | 0.746 | 1.009 | 0.301 | 1.588 |
| $\theta_r^P$ | 0.029 | 0.030 | 0.028 | 0.023 |
| $\theta_{\hat{\delta}}^P$ | −0.718 | −0.635 | 0.000 | −0.529 |
| $\theta_X^P$ | 3.120 | 4.498 | 5.946 | 6.159 |

assumption that the risk-free rate is an autonomous process seems appropriate). The coefficient $\alpha_X$ is significant across all commodities except for gold. It is high and positive for oil and copper, which is consistent with the theory of storage and indicates mean reversion in spot prices under the risk-neutral measure. The estimated $\alpha_X$ is lower for silver and negligible for gold, which is evidence against this source of mean reversion in these commodities. The sensitivity of convenience yields to interest rates $\alpha_r$ is significant and positive across commodities, which is consistent with the theory of storage.[28] Interestingly, it is higher for crude oil and copper than for gold and silver. Performing a likelihood ratio test to jointly test for the significance of $\alpha_r$ and $\alpha_X$, we find that they are highly significant for most commodities (though barely for gold at the 5% level—see Table III).

[28] As described in the introduction, most theoretical models do not allow for stochastic interest rates. However, assuming that costs of holding inventory increase with interest rates suggests a negative correlation between inventory and interest rates. We thus expect a positive relation between interest rates and convenience yield.

**Table V**

## Unconditional First and Second Moments with the Maximal Model Estimates

Unconditional first and second moment estimates from the maximal model of $\delta(t)$ and $X(t)$ for crude oil, copper, gold, and silver using weekly prices and interest rate data from January 2, 1990 to August 25, 2003.

| Unconditional Moments | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| $E[\delta]$ | 0.109 | 0.063 | 0.009 | 0.002 |
| $SD(\delta)$ | 0.210 | 0.116 | 0.010 | 0.018 |
| $E[X]$ | 3.120 | 4.498 | 5.946 | 6.159 |
| $SD(X)$ | 0.264 | 0.190 | 0.212 | 0.122 |
| $\mathrm{Corr}(\delta, X)$ | 0.790 | 0.776 | −0.246 | 0.475 |
| $E[e^X]$ | 23.45 | 91.45 | 390.91 | 476.46 |

The significance of the risk-premia parameters varies across commodities, but there are some consistent patterns. Most risk-premia coefficients related to the spot price (i.e., $\beta_{0X}$, $\beta_{XX}$) are significant.[29] In contrast, risk premia related to the interest rate dynamics are barely (or not) significant. Further, $\beta_{XX}$ is always negative implying that risk premia are time-varying and, in fact, negatively correlated with the spot price. All spot commodity prices exhibit mean reversion under the physical measure as evidenced by the positive coefficient of mean reversion $\alpha_X - \beta_{XX}$. When performing a likelihood-ratio test for the significance of time variation in risk premia (i.e., a joint test that all coefficients in the $\beta_{1Y}$ matrix are 0), we find that they are jointly significant (see Table III).

Overall, the results show that the maximal model, which allows convenience yields to be a function of the interest rate and spot price and which is associated with the more flexible time-varying risk-premia specification, significantly improves over nested models proposed in the literature. The joint likelihood-ratio tests of Table III suggest that allowing more general dynamics of the convenience yield is the more important feature. This may be due to the fact that spot price dependence in convenience yield results in mean reversion under both the risk-neutral and historical measures, whereas the time variation in risk premia only affects the strength of mean reversion under the physical measure.

We first provide more detailed discussions of the individual commodities, then summarize the implications for the dynamics of convenience yields and the sources of mean reversion in commodity spot prices, as well as the evidence on model (mis-)specification.

[29] For simplicity, in the estimation results presented we drop the time-varying risk premia parameters that have a $t$-ratio $< 1.0$, which corresponds to a level of significance of 31.7% (see also Duffee and Stanton (2002) for a discussion of the significance of essentially affine risk premia). This is the case of $\beta_{Xr}$ for oil, copper, and silver and $\beta_{X\hat{\delta}}$ for oil and gold.

### C.1. Crude Oil

We find that the oil price has a significant positive effect on the convenience yield ($\alpha_X = 0.248$). This implies strong mean reversion of log spot prices under the risk-neutral measure. Also, there is evidence of negative correlation between risk premia and spot prices. The parameter $\beta_{XX}$ is $-0.498$, implying that the mean reversion under the physical measure is higher than the mean reversion under the risk-neutral measure ($\kappa_X^P > \alpha_X$). The (historical) mean reversion in oil prices is due to both the convenience yield and the time variation in risk premia. The relation between the convenience yield and interest rates is significant and positive ($\alpha_r = 1.764$), which is consistent with the prediction of the theory of storage. All risk-neutral coefficients are significant for oil, except for some correlations, indicating that three factors are necessary to capture the dynamics of oil futures prices. The idiosyncratic component of the convenience yield, $\hat{\delta}$, has high volatility $\sigma_{\hat{\delta}} = 0.384$, has low persistence $\kappa_{\hat{\delta}} = 1.191$ and is positively correlated with the spot price $\rho_{\hat{\delta}X} = 0.795$. While this third factor is clearly a significant component of the convenience yield, it seems to be driven by innovations that are correlated with the spot market and that are short lived. The long-term maximal convenience yield is 0.109, which is the highest among the commodities studied (see Table V). Also from this table, the estimate for the long-term spot price is \$23.45 per barrel.

### C.2. Copper

Copper has a similar behavior as crude oil. We find a statistically significant positive relation between the spot price of copper and its convenience yield ($\alpha_X = 0.150$). This implies mean reversion in spot prices under the risk-neutral measure. We find a significant negative correlation between risk premia and spot prices ($\beta_{XX} = -0.859$). This implies that the mean reversion is stronger under the historical measure than under the risk-neutral measure. Table IV gives the point estimates of $\kappa_X^P = 1.009$ versus $\alpha_X = 0.150$ (in Table II). The relation between convenience yields and interest rates is positive and statistically significant as before. The idiosyncratic component of the convenience yield, $\hat{\delta}$, is quite volatile $\sigma_{\hat{\delta}} = 0.178$, not persistent $\kappa_{\hat{\delta}} = 1.048$, and positively correlated with the spot price $\rho_{\hat{\delta}X} = 0.588$. As for oil, the convenience yield in the copper market is primarily driven by both the spot price itself and economic factors that are correlated with spot price innovation and that are short lived.[30] Finally, Table V gives a long-term mean for the convenience yield of 0.063 and a long-term average copper price of ¢91.45 per pound.

### C.3. Gold

We find that there is a negligible relation between the convenience yield and gold spot prices ($\alpha_X = 0.000$). This suggests that there is no mean reversion in

[30] Of course, the convenience yield is also affected by the interest rate through the parameter $\alpha_r$ though to a lesser extent. Even though $\alpha_r$ is greater than $\alpha_X$, recall from equation (11) that the effect in the convenience yields is through the magnitude of $\alpha_r r(t)$ and $\alpha_X X(t)$.

gold prices under the risk-neutral measure. However, the price of gold exhibits mean reversion under the historical measure because of the negative correlation between the time-varying risk premia and the spot price ($\beta_{XX} = -0.301$). Interest rates seem to be more important in driving the convenience yield of gold than spot prices ($\alpha_r = 0.332$). The idiosyncratic factor has a small effect on the convenience yield. Its long-term mean is small ($\theta_{\hat{\delta}}^{Q} = -0.009$), its volatility is very low $\sigma_{\hat{\delta}} = 0.015$, it is somewhat persistent $\kappa_{\hat{\delta}}^{Q} = 0.392$, and it's not highly correlated with spot prices and interest rates ($\rho_{\hat{\delta}X} = 0.295$, $\rho_{r\hat{\delta}} = -0.047$). Overall, the convenience yield of gold is quite small, not very variable, and mainly driven by the interest rate. Table V shows that the convenience yield has a long-term mean of 0.009 and an unconditional standard deviation of only 0.010. The long-term price of gold is \$390.91 per troy ounce.

### *C.4. Silver*

The dynamics of silver share some characteristics with the behavior of gold. We find that silver has a low degree of mean reversion under the risk-neutral measure ($\alpha_X = 0.085$). Silver prices exhibit mean reversion under the historical measure due to the negative correlation between spot prices and risk premia ($\beta_{XX} = -1.503$). Interest rates have an effect on convenience yields similar to gold ($\alpha_r = 0.326$). The low volatility of the idiosyncratic factor ($\sigma_{\hat{\delta}} = 0.067$) suggests that the convenience yield of silver is mainly driven by spot prices and interest rates. The idiosyncratic factor, $\hat{\delta}$, follows a mean-averting process under the risk-neutral measure, which, due to its magnitude ($|\kappa_{\hat{\delta}}| > |\alpha_X|$), also induces mean aversion in the convenience yield. Table V shows that the long-term convenience yield is 0.002 and the unconditional standard deviation is 0.018. Finally, Table V gives a long-term average silver price of ¢476.46 per troy ounce.

### *C.5. Misspecification*

Since we estimate the parameters for each commodity separately, we obtain four different estimates for interest rate parameters. In general, the estimates seem reasonable (e.g., in line with estimates of single-factor models found in the literature) and do not vary significantly across estimation except for gold, which is weak evidence that the model correctly captures the relation among interest rates, convenience yield, and commodity prices. Not surprisingly, the autocorrelation coefficient for the term structure "measurement errors" is quite high ($\approx$0.99), indicating that at least a second factor is needed to capture the dynamics of the term structure. This is well known (Litterman and Scheinkman (1991)), but our primary focus is to analyze the term structure of commodity futures and we expect an additional term structure factor to have only limited explanatory power for commodity prices. More important for our study are the "measurement errors" for the commodity futures. The autocorrelation coefficients are lower than for interest rates (0.78 on average), but significant. Figure 4 graphs time series of the pricing errors of some futures contracts for the

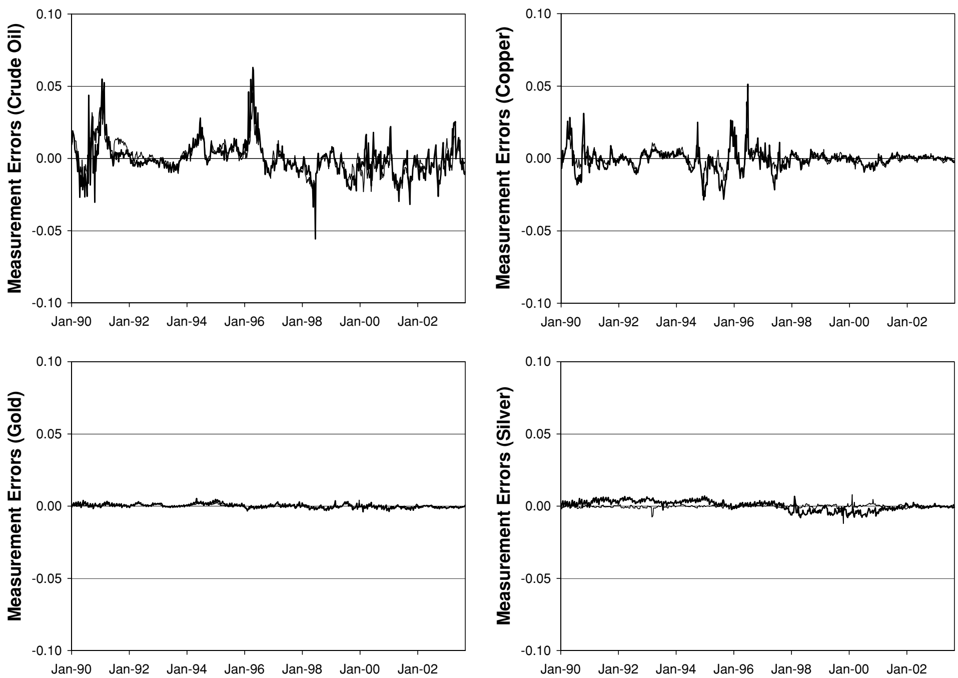


**Figure 4. Pricing errors.** Difference between true and estimated (log) futures prices using the maximal model for the period from January 2, 1990 to August 25, 2003. The thick line and the thin line correspond to the F01 and F18 contracts, respectively.

**Table VI**
## Statistics of Pricing Errors Using the Maximal Model

Mis-specification statistics for the error terms of the F01 futures contract using the maximal model. The error term $u_t$ is $\log(F01) - \log(\widetilde{F01})$, where $\widetilde{F01}$ is the estimated F01 futures contract. The statistics are for the four commodities using data from January 2, 1990 to August 25, 2003.

| Error | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| Mean$u_t$ | 0.000 | −0.001 | 0.000 | 0.000 |
| SD$u_t$ | 0.012 | 0.008 | 0.001 | 0.003 |
| Max$u_t$ | 0.063 | 0.051 | 0.005 | 0.008 |

four commodities. In Table VI, we present some summary statistics about these pricing errors for the maximal model. There does not seem to be a systematic bias in the fit of the model. Not surprisingly, the analysis of the unconditional pricing errors ($u_t$) shows that the model performs (in terms of MSE) slightly less well with the two commodities that exhibit higher volatility (i.e., oil and copper).

Inspection of the time series of futures prices indicates that perhaps some of these errors are attributable to the inability of the pure-diffusion Gaussian model to accommodate the presence of jumps in spot prices. For example, gold prices experienced a +25% jump in prices during September–October 1999. This jump followed an announcement made by the European central banks in response to increased pressures of gold producers to cut sales of gold reserves. Further, demand for gold at that time may have been fueled by the Y2K uncertainty.

To make sure that the presence of jumps in the spot time series does not affect our empirical findings, we reestimate the model by allowing for jumps in the underlying spot price dynamics.

### *D. Estimation of the Jump Component in Commodity Spot Prices*

We allow for jumps in commodity prices by considering the model introduced in Proposition 2, where the dynamics of $X(t)$ are modified as follows:

$$dX(t) = \left( r(t) - \delta(t) - \frac{1}{2}\sigma_X^2 - \sum_{i=1}^{3}(\varphi_i - 1)\lambda_i \right) dt + \sigma_X \, dZ_X^Q(t) + \sum_{i=1}^{3} \upsilon_i(t)\, dN_i(t), \tag{36}$$

where $N_i(t) = \sum_j \mathbf{1}_{\{\tau_i^j \le t\}}$ is the counting process associated with a sequence of stopping times $\tau_i^1, \tau_i^2, \ldots$ generated by a standard Poisson process with $Q$-measure intensity $\lambda_i^Q$ (see Brémaud (1981) for a rigorous exposition of point processes). The $\upsilon_i(\tau_i^j)\, \forall j = 1, 2 \ldots$ are i.i.d. random variables that are independent of both the Poisson process and the Brownian motions. Further, we assume $\upsilon_1$ is Gaussian with mean jump size $m_1$ and standard deviation $v_1$, while $\upsilon_2$ and $\upsilon_3$ have constant jump sizes $m_2$ and $m_3$, respectively (i.e., $v_2 = v_3 = 0$).[31] We denote the Laplace transform of the random variable $\upsilon_i$ by $\varphi_i = e^{m_i + \frac{v_i^2}{2}}, i = 1, \ldots, 3$.

Applying Itô's lemma to the spot price defined as before by $S(t) = e^{X(t)}$, we obtain

$$\frac{dS(t)}{S(t^-)} = (r(t) - \delta(t))\, dt + \sigma_X \, dZ_X^Q(t) + \sum_i dM_i^Q(t), \tag{37}$$

where $M_i^Q(t) := \int_0^t (e^{\upsilon_i(s)} - 1)\, dN_i(s) - (\varphi_i - 1)\lambda_i^Q t$ is a $Q$-Martingale. Thus,

$$E_t^Q \left[ \frac{dS(t)}{S(t^-)} \right] = (r(t) - \delta(t))\, dt$$

and, as before, $\delta$ retains the interpretation of a convenience yield that accrues to the holder of the commodity similarly to a dividend yield.

To empirically implement the model, we need a specification of risk premia for both Brownian motion and jump risk. We use the same "essentially affine" risk-premium structure for Brownian motions as in equation (24). We also study the

[31] We find that allowing for more than one jump to have a stochastic jump size does not improve the likelihood and thus we choose to report only the constant jump size case.

risk premia for the jump intensities. If jump risk is systematic (e.g., if there is a common jump in the pricing kernel), then intensities need to be risk adjusted. If jump risk is nonsystematic (e.g., because it is conditionally diversifiable as in Jarrow, Lando, and Yu (2003) or "extraneous" as in Collin-Dufresne and Hugonnier (1999)) then intensities are not risk adjusted and remain the same under both measures. We empirically find that allowing intensities to change does not improve the fit of the model significantly.[32] We report the case in which the jump intensities are not risk adjusted, that is, $N_i$ are Poisson processes with the same intensities under both measures. Further, we assume jump-size risk is not priced, that is, that the jump distribution is the same under both measures.[33]

Following DK (1996), futures prices may be computed in closed form for this Gaussian jump-diffusion model. We report the closed-form formulas in Appendix F. We estimate the model using maximum likelihood as indicated in Section III.B. The only change is that the transition density of the log spot price is no longer Gaussian. Following Ball and Torous (1983), Jorion (1988), and Das (2002), we approximate the transition density by a mixture of Gaussian distributions (the approximation would be exact if the time interval was infinitesimal). Several problems arise when implementing this approach. Mainly, the likelihood function is unbounded if the model is estimated without any restrictions. We use Honoré's (1998) approach to obtain consistent estimates of the parameters. More details about the estimation procedure and approximation to the likelihood function are presented in Appendix G.

Results for the parameter estimates are reported in Table VII.[34] For all commodities, we find significant evidence for the presence of a frequent stochastic jump component with mean $m_1$ close to 0. This reflects small, variable, and frequent jumps in the spot price that are unaccounted for by the pure-diffusion model.[35] For all commodities, we also find evidence for the presence of positive and negative less-frequent jumps, except for gold where instead of a negative jump, we find a highly frequent small positive jump.[36] For copper and silver, the positive jump has a mean between 7.3% and 10%, and occurs twice every 2 years on average. For gold, this jump has a similar mean of $m_2 = 0.096$, but occurs only once every 6 years. For crude oil, this jump is not significant. The negative jumps have different means and intensities across commodities. For

[32] In fact, futures prices are very insensitive to the presence of these jumps (because they have almost 0 mean and futures prices are $Q$-expectations of the future spot price). As a result, most improvement in the likelihood is due to the improvements in the $P$-measure distribution for this jump component. See the discussion below and Appendix F.

[33] Obviously, for the constant jump size cases, this is a requirement since both measures must be equivalent.

[34] For the results presented, we keep jumps with parameter estimates that have a $t$-ratio $>1$.

[35] Small and frequent jumps may be suggestive of Levy processes; see, for example, Bakshi and Madan (2000).

[36] Note, however, that the average mean is negative and larger in magnitude than for the other commodities $m_1 = -0.017$. The larger positive jump found for gold may be related to the special September to November 1999 period discussed previously.

**Table VII**

## Maximum-Likelihood Parameter Estimates for the Triple-Jump Model

Maximum-likelihood parameter estimates for the triple-jump model defined in equation (36) in the paper for crude oil, copper, gold, and silver weekly prices and interest rate data from January 2, 1990 to August 25, 2003.

| Parameter | Crude Oil Estimate (SE) | Copper Estimate (SE) | Gold Estimate (SE) | Silver Estimate (SE) |
|---|---|---|---|---|
| $\kappa_r^Q$ | 0.027 | 0.035 | 0.031 | 0.034 |
| | (0.007) | (0.007) | (0.007) | (0.007) |
| $\kappa_{\hat{\delta}}^Q$ | 1.190 | 1.046 | 0.384 | −0.173 |
| | (0.023) | (0.038) | (0.035) | (0.007) |
| $\alpha_r$ | 1.772 | 0.845 | 0.324 | 0.264 |
| | (0.083) | (0.095) | (0.049) | (0.069) |
| $\alpha_X$ | 0.250 | 0.145 | 0.003 | 0.131 |
| | (0.010) | (0.015) | (0.003) | (0.006) |
| $\theta_r^Q$ | 0.056 | 0.118 | 0.096 | 0.116 |
| | (0.028) | (0.015) | (0.017) | (0.016) |
| $\theta_{\hat{\delta}}^Q$ | −0.840 | −0.656 | −0.025 | −0.815 |
| | (0.033) | (0.062) | (0.015) | (0.037) |
| $\beta_{0r}$ | 0.004 | 0.002 | 0.002 | −0.001 |
| | (0.009) | (0.009) | (0.009) | (0.011) |
| $\beta_{0\hat{\delta}}$ | 1.245 | −0.411 | −0.010 | −1.029 |
| | (0.324) | (0.333) | (0.016) | (0.525) |
| $\beta_{0X}$ | 0.024 | 3.063 | −1.100 | 12.070 |
| | (0.489) | (1.920) | (1.309) | (6.272) |
| $\beta_{rr}$ | 0.146 | −0.170 | −0.137 | −0.111 |
| | (0.169) | (0.167) | (0.172) | (0.209) |
| $\beta_{\hat{\delta}\hat{\delta}}$ | −1.914 | −0.733 | −1.003 | −1.265 |
| | (0.405) | (0.522) | (0.449) | (0.671) |
| $\beta_{Xr}$ | | | −5.677 | |
| | | | (2.413) | |
| $\beta_{X\hat{\delta}}$ | | 1.365 | | 6.767 |
| | | (0.846) | | (5.960) |
| $\beta_{XX}$ | 0.054 | −0.483 | 0.233 | −1.065 |
| | (0.165) | (0.340) | (0.232) | (0.395) |
| $\sigma_r$ | 0.009 | 0.009 | 0.009 | 0.009 |
| | (0.000) | (0.000) | (0.000) | (0.000) |
| $\sigma_{\hat{\delta}}$ | 0.384 | 0.178 | 0.015 | 0.024 |
| | (0.013) | (0.006) | (0.001) | (0.001) |
| $\sigma_X$ | 0.319 | 0.173 | 0.080 | 0.208 |
| | (0.014) | (0.019) | (0.007) | (0.008) |
| $\rho_{\hat{\delta}X}$ | 0.957 | 0.810 | 0.363 | −0.957 |
| | (0.017) | (0.081) | (0.067) | (0.024) |
| $\rho_{r\hat{\delta}}$ | −0.009 | 0.106 | −0.039 | 0.007 |
| | (0.039) | (0.038) | (0.054) | (0.041) |
| $\rho_{rX}$ | 0.060 | 0.184 | −0.120 | 0.046 |
| | (0.043) | (0.048) | (0.053) | (0.041) |
| $m_1$ | 0.001 | −0.002 | −0.017 | 0.000 |
| | (0.001) | (0.003) | (0.014) | (0.002) |

(*continued*)

**Table VII**—*Continued*

| Parameter | Crude Oil Estimate (SE) | Copper Estimate (SE) | Gold Estimate (SE) | Silver Estimate (SE) |
|---|---|---|---|---|
| $v_1$ | 0.025 | 0.017 | 0.019 | 0.016 |
| | (0.006) | (0.012) | (0.012) | (0.006) |
| $\lambda_1$ | 65.126 | 63.829 | 8.800 | 49.869 |
| | (19.231) | (51.522) | (6.115) | (21.485) |
| $m_2$ | | 0.073 | 0.096 | 0.100 |
| | | (0.013) | (0.009) | (0.010) |
| $\lambda_2$ | | 0.674 | 0.156 | 0.755 |
| | | (0.541) | (0.113) | (0.295) |
| $m_3$ | −0.176 | −0.083 | 0.022 | −0.115 |
| | (0.015) | (0.021) | (0.003) | (0.008) |
| $\lambda_3$ | 0.154 | 0.219 | 8.063 | 0.369 |
| | (0.093) | (0.317) | (3.184) | (0.187) |
| $\rho_F$ | 0.796 | 0.699 | 0.813 | 0.811 |
| | (0.011) | (0.013) | (0.011) | (0.010) |
| $\rho_P$ | 0.993 | 0.986 | 0.989 | 0.986 |
| | (0.003) | (0.003) | (0.003) | (0.003) |
| Log-likelihood | 57,210.5 | 51,782.4 | 72,952.2 | 66,216.7 |

**Table VIII**

## Likelihood Ratio Tests for the Triple-Jump Model

Likelihood ratio for the triple-jump model and the maximal model. The 5% significance level for the jumps constraints is given by Prob$\{\chi^2_7 \geq 14.07\} = 0.05$.

| Restriction | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| $m_i = v_i = \lambda_i = 0$ | 114.08 | 41.902 | 104.898 | 203.721 |

crude oil, the jump size is $m_3 = -0.176$ and is very infrequent (once every 6 years on average). For copper and silver, these jumps have means of −8.3% and −11.5%, and they occur, on average, every 5 and 3 years, respectively. In Table VIII we carry out a likelihood-ratio test for the hypothesis of having no jumps, which is clearly rejected. The inclusion of jumps appears to be especially significant for crude oil and silver.

Overall, however, our previous results seem mostly robust to the inclusion of jumps. When we compare the parameter estimates to those obtained without jumps in Table II, we see that the estimates of risk-neutral drift parameters are almost unchanged. Including jumps mainly affects the estimates of the volatility coefficients and the risk-premia parameters. These results can be explained by the fact that jumps in the spot price have little impact on the predicted cross section of futures prices. Indeed, we show in Appendix F that futures prices are affected by the presence of jumps only if $\alpha_X \neq 0$. The intuition

**Table IX**

**Correlation Matrix of Log Commodity Prices and Interest Rates**

Correlation matrix for the weekly changes in log commodity prices and interest rate data from January 2, 1990 to August 25, 2003.

| | Crude Oil | Copper | Gold | Silver | Interest Rate |
|---|---|---|---|---|---|
| Crude oil | 1.00 | 0.06 | 0.18 | 0.14 | 0.04 |
| Copper | 0.06 | 1.00 | 0.15 | 0.20 | 0.12 |
| Gold | 0.18 | 0.15 | 1.00 | 0.60 | −0.04 |
| Silver | 0.14 | 0.20 | 0.60 | 1.00 | −0.01 |
| Interest rate | 0.04 | 0.12 | −0.04 | −0.01 | 1.00 |

is that futures prices are martingales under the risk-neutral measure. The no-arbitrage restriction on the risk-neutral drift of the spot price (i.e., equation (4)) implies that jumps in the spot price can only "matter" if there is a common jump in the spot rate and/or the convenience yield.[37] Further, we also show in the appendix that for the estimated jump intensity and jump distribution, the impact of jumps on futures prices is negligible. However, accounting for jumps helps better capture the historical measure dynamics of futures prices.

### *E. Restrictions on Common Pricing Kernel Dynamics*

In perfectly integrated markets, all commodities should be priced by the same pricing kernel. One may thus wonder whether the specification of risk premia proposed in Section II is consistent with some arbitrage-free dynamics of a (common) pricing kernel, and whether these implied dynamics are economically reasonable. An alternative to our implementation might be to propose a joint specification of the dynamics of the pricing kernel and all commodity prices and perform a joint estimation.[38] The latter would seem particularly appropriate if commodity prices are largely driven by a small set of common factors. A simple look at the correlation structure (Table IX) shows that, except for gold and silver, innovations in commodity prices exhibit low correlation. A principal component analysis reveals that the factors driving oil and interest rates are distinct from factors driving the metal prices (see Table X). Indeed, the first eigenvector loads almost exclusively on oil, and the last eigenvector exclusively on the risk-free rate. Further, the eigenvalues corresponding to the three metal factors are of the same order of magnitude (the first two metal factors represent around 16% of the total variance and the third about 4%), which suggests that there is not a predominant common factor in the metal market.

This suggests that little is lost in performing the estimation separately for oil and metals. However, there may be some scope to perform a common estimation

[37] Hilliard and Reis (1998), for example, find that in their model, jumps have no impact on futures prices. Their convenience yield model is not maximal, however.

[38] We thank a referee for making this point.

**Table X**
**Principal Component Analysis of Log Commodity Prices and Interest Rates**

Principal component decomposition for the weekly changes in log prices and interest rate data from January 2, 1990 to August 25, 2003.

| | PC(1) | PC(2) | PC(3) | PC(4) | PC(5) |
|---|---|---|---|---|---|
| Crude oil | 0.99 | −0.16 | −0.05 | 0.03 | 0.00 |
| Copper | 0.07 | 0.63 | −0.77 | 0.00 | 0.00 |
| Gold | 0.08 | 0.30 | 0.25 | −0.92 | 0.00 |
| Silver | 0.13 | 0.70 | 0.58 | 0.40 | 0.00 |
| Interest rate | 0.00 | 0.00 | 0.00 | 0.00 | −1.00 |
| Eigenvalue | 0.0030 | 0.0012 | 0.0008 | 0.0002 | $1.05 \times 10^{-6}$ |
| % explained | 58.11% | 22.93% | 15.43% | 3.51% | 0.02% |

of the pricing kernel dynamics and all three metal prices. Fortunately, it is possible to test whether the restrictions imposed by our specification of risk premia are significant.

Indeed, assume that there exists a filtered probability space $(\Omega, \mathcal{F}, P)$, where the filtration $\mathcal{F}$ is the natural filtration generated by two $n$-dimensional vectors of Brownian motions $B^P(t)$ and $Z^P(t)$. Consider the following dynamics of log spot commodity prices ($X_t^i$ $i = 1, \ldots, n$) and the common pricing kernel $M_t$:

$$\frac{dM(t)}{M(t)} = -r\,dt - \sum_{i=1}^{n} \frac{1}{\sigma_i}(\beta_{0i} + \beta_{1i} X_i(t))\,dB_i^P(t), \tag{38}$$

$$dX_i(t) = \mu_i(t)\,dt + \sigma_i\,dZ_i^P(t), \tag{39}$$

$$= \left(r - \delta_i - \frac{\sigma_i^2}{2}\right) dt + \sigma_i\,dZ_i^Q(t). \tag{40}$$

The last equality is the standard absence of arbitrage restriction, where the $Z_i^Q(t)$ $i = 1, \ldots, n$ are Brownian motions under the risk-neutral measure $Q$, which is equivalent to $P$ and defined by $\frac{dQ}{dP}|_T = e^{rT}\frac{M(T)}{M(0)}$.[39] Note that for simplicity of notation, we assume here that the risk-free rate and the convenience yields are constant, but our argument extends straightforwardly to the more general case. By Girsanov's theorem, we have (defining $dB_i(t)\,dZ_j(t) = \eta_{ij}\,dt$)

$$Z_i^Q(t) = Z_i^P(t) + \sum_{j=1}^{n} \int_0^t \eta_{ij} \frac{1}{\sigma_j}(\beta_{0j} + \beta_{1j} X_j(s))\,ds \quad \forall i = 1, \ldots, n, \tag{41}$$

[39] To ensure that this is an appropriate change of measure, we need to verify some additional regularity conditions. In this Gaussian framework, this follows straightforwardly from Theorem 7.15 in Liptser and Shiryaev (1977, p. 279).

and thus the expected change in log spot prices is given by

$$\mu_i(t) = r - \delta_i - \frac{\sigma_i^2}{2} + \sum_{j=1}^{n} \eta_{ij}(\beta_{0j} + \beta_{1j} X_j(t)). \tag{42}$$

This framework allows us to see the implicit restrictions put by our specification of risk premia on the joint dynamics of the pricing kernel and spot prices. We are basically restricting the correlation structure of the vector of Brownian motions. Specifically, comparing equations (24) and (41), we see that our risk-premium specification of Section II is consistent with the common pricing kernel model above if and only if $\eta_{ij} = 0\, \forall i \neq j$. Fortunately, we can test this restriction without resorting to full-fledged joint estimation (which would be highly computationally intensive). Comparing equations (39), (40), and (42) with Proposition 2 and equation (3), we see that the only implication of the more general model specification (i.e., with $\eta_{ij} \neq 0$ for some $i \neq j$) for our data set is that the expected return of commodity $i$ (say oil) can depend on the log price of commodity $j$ (say gold). We can easily test this by performing a vector-auto-regression (VAR) and doing a likelihood-ratio test to see whether allowing for cross-dependence in the expected changes of commodities is significant.

In Tables XI we present the results for the following VAR: $X_t = c + \phi X_{t-1} + \mu_t$, where $\mu_t$ follows an AR(1) process $\mu_t = \rho\mu_{t-1} + \varepsilon_t$ and $\varepsilon_t \sim N(0, \Sigma)$. The components of $X_i(t)$ for $i = 1, \ldots, 5$ are the (log) prices for crude oil, copper, gold, silver, and the 6-month interest rate. Table XII presents the results for the same autoregression, but keeping only copper, gold, and silver, since the factor analysis above suggests that this is where a joint estimation should benefit most. The results show that almost all of the off-diagonal terms $\phi_{ij}$ with $i \neq j$ are not significant. Further, in both cases the likelihood-ratio test cannot reject the hypotheses that all of the $\phi_{ij}$ with $i \neq j$ are 0. In other words, the results from the VAR support our specification assumption that $\eta_{ij} = 0\, \forall i \neq j$. Given the overall low correlations among various commodities (i.e., $dZ_i\, dZ_j$) documented in Table IX, this suggests that there would be little to gain from performing a joint estimation with all commodity prices at once (this seems especially true, given the size of our data set).

### *F. Summary of the Results*

*Implied Convenience Yields.* In Figure 5, we present the implied convenience yields for the four commodities. These graphs are obtained using the estimated $\{r, \hat{\delta}, X\}$ state variables and then calculating the implied convenience yield for each time-series observation.[40] The figure clearly distinguishes oil and copper,

[40] We have presented the four implied convenience yields with the same scale for comparison purposes. The magnitude of the convenience yield on oil is on a few occasions as high as 50% (which seems large). We verified that this occurs on dates where the backwardation in the futures curve was correspondingly high. In fact, the convenience yield backed out from the two shortest maturity futures contracts assuming a Black model was even higher (reflecting a highly negative slope in the short-term futures curve).

**Table XI**

## Vector Autoregression to Test Restrictions in Common Pricing Kernel

Vector autoregression (VAR) to test the restrictions imposed by our risk-premium specification on the correlation structure $(dB_i(t)\,dB_j(t))$. The VAR is $X_t = c + \phi X_{t-1} + \mu_t$, where $\mu_t$ follows an AR(1) process $\mu_t = \rho\mu_{t-1} + \varepsilon_t$ and $\varepsilon_t \sim N(0, \Sigma)$. Here, $X_i(t)$ for $i = 1, \ldots, 4$ are the (log) prices for crude oil, copper, gold, and silver, respectively, and $X_5(t)$ is the time series for the 6-month interest rate. The results are for weekly prices and interest rate data from January 2, 1990 to August 25, 2003. The likelihood-ratio tests the hypothesis that all off-diagonal terms of the $\phi$ matrix are 0. The 5% significance level for this constraint is given by $\text{Prob}\{\chi^2_{20} \geq 31.41\} = 0.05$.

| | Unrestricted Correlation Structure | | Restricted Correlation Structure | |
|---|---|---|---|---|
| Parameter | Estimate | (SE) | Estimate | (SE) |
| $c_1$ | 0.123 | (0.076) | 0.046 | (0.023) |
| $c_2$ | −0.058 | (0.111) | 0.038 | (0.032) |
| $c_3$ | 0.141 | (0.085) | 0.038 | (0.025) |
| $c_4$ | 0.341 | (0.072) | 0.140 | (0.048) |
| $c_5$ | −0.011 | (0.003) | 0.000 | (0.000) |
| $\phi_{11}$ | 0.982 | (0.008) | 0.985 | (0.007) |
| $\phi_{12}$ | 0.017 | (0.018) | | |
| $\phi_{13}$ | −0.031 | (0.023) | | |
| $\phi_{14}$ | 0.007 | (0.011) | | |
| $\phi_{15}$ | −0.121 | (0.247) | | |
| $\phi_{21}$ | 0.000 | (0.011) | | |
| $\phi_{22}$ | 0.961 | (0.012) | 0.992 | (0.007) |
| $\phi_{23}$ | 0.043 | (0.014) | | |
| $\phi_{24}$ | −0.005 | (0.016) | | |
| $\phi_{25}$ | 0.229 | (0.166) | | |
| $\phi_{31}$ | −0.003 | (0.008) | | |
| $\phi_{32}$ | 0.007 | (0.013) | | |
| $\phi_{33}$ | 0.983 | (0.013) | 0.993 | (0.004) |
| $\phi_{34}$ | −0.010 | (0.011) | | |
| $\phi_{35}$ | −0.086 | (0.164) | | |
| $\phi_{41}$ | −0.011 | (0.009) | | |
| $\phi_{42}$ | −0.004 | (0.006) | | |
| $\phi_{43}$ | −0.007 | (0.008) | | |
| $\phi_{44}$ | 0.960 | (0.011) | 0.977 | (0.008) |
| $\phi_{45}$ | 0.015 | (0.036) | | |
| $\phi_{51}$ | 0.000 | (0.000) | | |
| $\phi_{52}$ | 0.000 | (0.000) | | |
| $\phi_{53}$ | 0.001 | (0.000) | | |
| $\phi_{54}$ | 0.001 | (0.000) | | |
| $\phi_{55}$ | 0.995 | (0.003) | 0.998 | (0.002) |
| $\rho_1$ | −0.155 | (0.046) | −0.154 | (0.038) |
| $\rho_2$ | −0.031 | (0.048) | −0.040 | (0.061) |
| $\rho_3$ | 0.035 | (0.066) | 0.032 | (0.027) |
| $\rho_4$ | 0.025 | (0.041) | 0.020 | (0.032) |
| $\rho_5$ | −0.021 | (0.031) | 0.013 | (0.021) |
| Log-likelihood | 18.57 | | 18.53 | |
| LR test | 0.08 | | | |

**Table XII**

## VAR to Test Restrictions in Common Pricing Kernel of Copper, Gold, and Silver

Vector autoregression (VAR) to test the restrictions imposed by our risk-premium specification on the correlation structure $(dB_i(t)\,dB_j(t))$. The VAR is $X_t = c + \phi X_{t-1} + \mu_t$, where $\mu_t$ follows an AR(1) process $\mu_t = \rho\mu_{t-1} + \varepsilon_t$ and $\varepsilon_t \sim N(0, \Sigma)$. Here, $X_i(t)$ for $i = 2, \ldots, 4$ are the (log) prices for copper, gold, and silver, respectively. The results are for weekly prices and interest rate data from January 2, 1990 to August 25, 2003. The likelihood ratio tests the hypothesis that all off-diagonal terms of the $\phi$ matrix are 0. The 5% significance level for this constraint is given by Prob$\{\chi^2_6 \geq 12.59\} = 0.05$.

| | Unrestricted Correlation Structure | | Restricted Correlation Structure | |
|---|---|---|---|---|
| Parameter | Estimate | (SE) | Estimate | (SE) |
| $c_2$ | −0.100 | (0.083) | 0.037 | (0.033) |
| $c_3$ | 0.138 | (0.057) | 0.044 | (0.023) |
| $c_4$ | 0.247 | (0.088) | 0.136 | (0.048) |
| $\phi_{22}$ | 0.978 | (0.008) | 0.992 | (0.007) |
| $\phi_{23}$ | 0.030 | (0.013) | | |
| $\phi_{24}$ | 0.005 | (0.009) | | |
| $\phi_{32}$ | 0.001 | (0.005) | | |
| $\phi_{33}$ | 0.989 | (0.008) | 0.992 | (0.004) |
| $\phi_{34}$ | −0.012 | (0.007) | | |
| $\phi_{42}$ | −0.004 | (0.007) | | |
| $\phi_{43}$ | −0.004 | (0.012) | | |
| $\phi_{44}$ | 0.967 | (0.010) | 0.978 | (0.008) |
| $\rho_2$ | −0.043 | (0.039) | −0.043 | (0.038) |
| $\rho_3$ | 0.039 | (0.034) | 0.031 | (0.031) |
| $\rho_4$ | 0.022 | (0.027) | 0.019 | (0.059) |
| Log-likelihood | 9.72 | | 9.71 | |
| LR test | 0.02 | | | |

which have highly volatile implied convenience yields, from silver and gold, whose convenience yields are close to 0 and exhibit little variability. This is attributable, in part, to a higher standard deviation of the spot commodity prices for oil and copper, as well as a higher volatility of the residual third factor, $\sigma_{\hat{\delta}}$ (see Table II). Table V confirms these results. Gold and silver have implied convenience yields of about 0.9% and 0.2%, whereas copper and oil have convenience yields of, respectively, 6.3% and 10.9%.

*Sources of Mean-Reversion: Convenience Yield and Time Varying Risk Premia.* Overall, our results suggest that the maximal convenience yield model improves upon all nested specifications tested in the literature (such as the models studied by Schwartz (1997)). We find that the price level dependence in convenience yield is significant and higher for assets that tend to be used as inputs to production, such as oil and copper. It is also significant for silver. Time variation in risk premia, on the other hand, seems to be highest for assets that also may serve as a store of value and thus, perhaps, more closely resemble financial

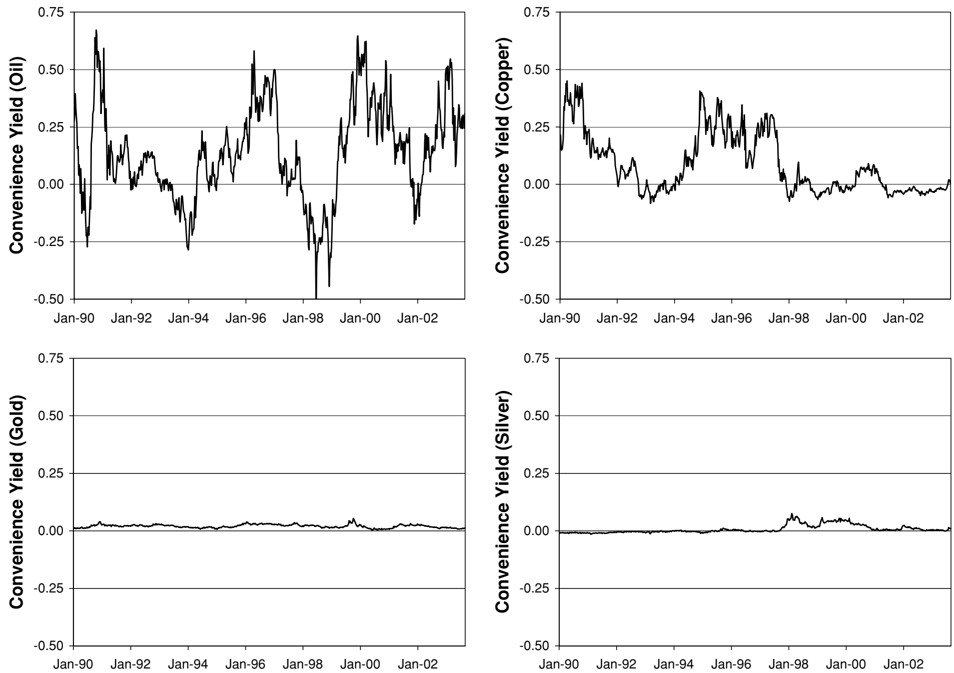


**Figure 5. Maximal convenience yield.** Implied convenience yield from the maximal model for crude oil, copper, gold, and silver from January 2, 1990 to August 25, 2003.

assets such as gold and silver. Our results show that both convenience yields as justified by the option/storage theoretic models (Litzenberger and Rabinowitz (1995), Deaton and Laroque (1992), RSS (2000), and Casassus, Collin-Dufresne, and Routledge (2003)), and time-varying risk premia (e.g., Fama and French (1987, 1988)) contribute to explaining mean reversion in commodity prices with more or less impact depending on the nature of the commodity.

Aside from their econometric interest, these results also have economic implications. In the following section, we offer two simple applications that demonstrate the impact on valuation and risk management of ignoring the various sources of mean reversion in commodity prices.

## IV. Implication of Mean Reversion for Option Pricing and Value-at-Risk

Schwartz (1997) shows that the stochastic behavior of commodity prices may have important implications for the valuation of commodity-related securities. We have documented that allowing convenience yields to be a function of spot prices and interest rates, and allowing risk premia to be time varying, better captures the dynamics of commodity futures prices. Both features have largely been ignored by previous commodity pricing models. We focus on two simple examples to demonstrate how significant the implications are for economic applications, namely: (i) valuation of options, and (ii) computation of value at risk.

### *A. Option Pricing*

As discussed previously, allowing convenience yields to be a function of the spot price effectively induces mean reversion under the risk-neutral measure. Since the latter matters for valuation, we expect this to affect the cross section of option prices. Using the Fourier inversion approach introduced by Heston (1993), we can compute in closed form (up to a Fourier transform inversion) European option values within our three-factor affine framework.[41] We compute the option value using two sets of parameters. First, we use the parameters corresponding to our maximal convenience yield model as given in Table II. Second, we reestimate the parameters assuming one were to ignore the level dependence in convenience yields (i.e., setting $\alpha_X = \alpha_r = 0$ in our model). This basically corresponds to estimating model 3 of Schwartz (1997), but with a more flexible specification of risk premia and an AR(1) representation of the measurement errors.

The option prices obtained for each commodity with the two sets of parameters are shown in Figure 6. For each commodity, we value European call options written on a unit of the asset with a maturity of 2 years, and strike prices of \$25 per barrel for oil, ¢100 per pound for copper, \$350 per troy ounce for gold, and ¢550 per troy ounce for silver. The figure shows that the difference can become quite important for oil, copper, and silver, especially for options that are at- and in-the-money. For gold, the difference in option values is small, indicating that the coefficient $\alpha_X$, while statistically significant, has a small economic impact.

For commodities with a positive and significant relation between convenience yields and spot prices (i.e., crude oil, copper, and silver), ignoring the level dependence in convenience yields, leads to overestimation of call option values. The direction of the bias for a positive $\alpha_X$ is expected for two reasons. First, the maximal convenience yield model effectively introduces mean reversion under the risk-neutral measure and thus leads to reduced term volatility, which in turn reduces option prices. Second, a positive $\alpha_X$ implies a convenience yield which is stochastic and increasing in the spot price. This contributes to decreasing call option prices for in-the-money options.[42]

The size of the error for the estimated parameters is quite dramatic. For example, the error is close to 30% for in-the-money options written on crude oil. This suggests that appropriately modeling the dynamics of convenience yields may have important consequences for investment decisions within real-option models. For natural resource investments related to commodities such

[41] See DPS (2000) for a thorough exposition of this option valuation approach.

[42] This intuition is the analog to a call option on a dividend-paying stock in the Black-Scholes world. The higher the dividend rate, the lower the price of the option. Note, however, that the "naïve" model, which assumes $\alpha_r = \alpha_X = 0$, since it is reestimated, gets the average level of convenience yield right.

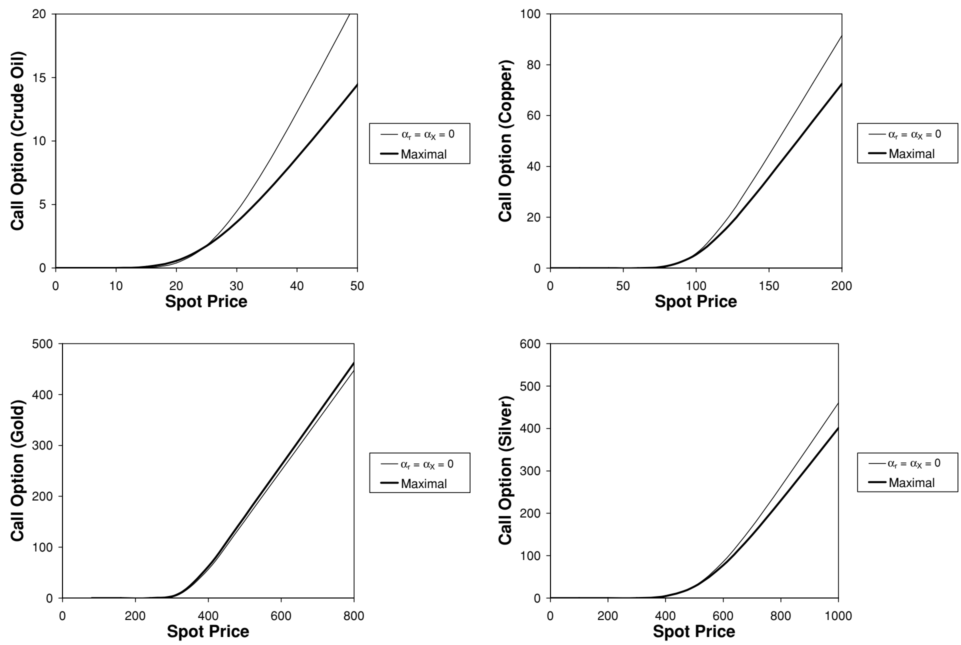


**Figure 6. Option prices.** Two-year maturity European call option prices using the maximal model. The strike price for oil is $25 per barrel, for copper is ¢100 per pound, for gold is $350 per troy ounce, and for silver is ¢550 per troy ounce. Spot and options prices are in the same unit as strike prices. Each line corresponds to a different set of parameters. The bold line corresponds to the maximal model, while the thin line assumes $\alpha_r = \alpha_X = 0$ (and reestimation of parameters).

as crude and copper, our results suggest that in a typical waiting to "invest" (Majd and Pindyck (1987)) framework that ignores the spot price dependence in convenience yields, the optimal investment rule would have a tendency to postpone investment suboptimally.

### *B. Value-at-Risk*

Our results indicate that for metals such as gold, silver, and copper, a substantial part of the mean reversion in spot prices is due to negative covariation between spot prices and risk premia. This implies that ignoring the time variation of risk premia during estimation will lead to misestimation of the holding period return distribution of commodities. To illustrate the latter for each commodity, we compute the value-at-risk (VAR) of the 5-year return on a portfolio invested in one unit of the commodity. By definition, the VAR is computed from the total (i.e., cum dividend) return under the historical measure. It thus allows us to focus on the effect of time-varying risk premia.

As before, we compute the VAR for two different sets of parameters. One corresponds to our maximal model, that is, Table II. For the other, we reestimate the three-factor model, but constraining risk premia to be constant (i.e.,

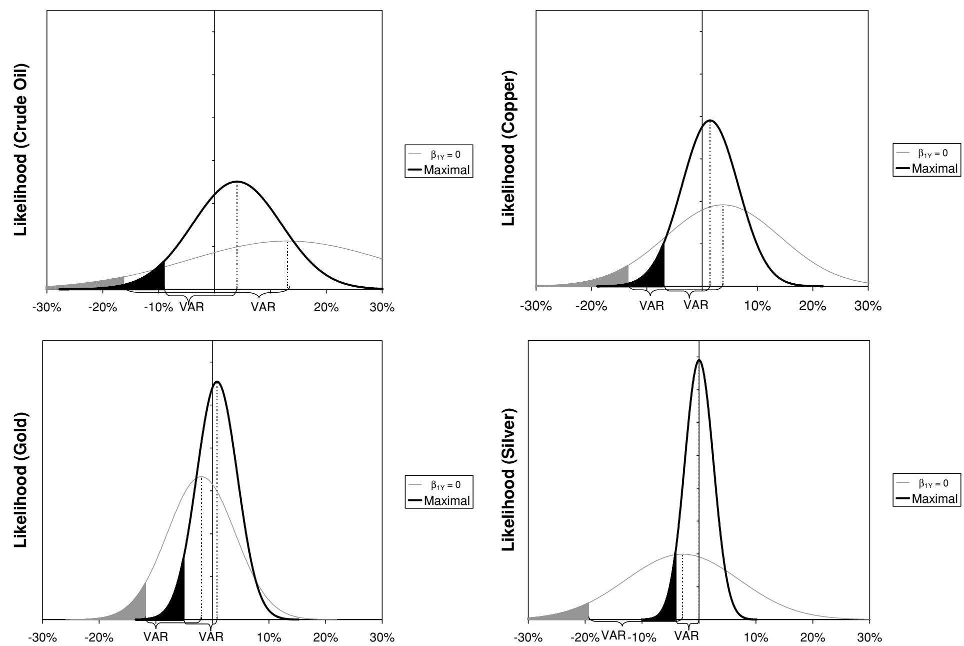


**Figure 7. Value-at-risk.** Distribution of returns and value-at-risk for holding one unit of commodity for 5 years using the maximal model. Value-at-risk is calculated at a 5% significance level. The two distributions correspond to a different set of parameters. The bold line uses copper estimates from the maximal model, while the thin line assumes constant risk premia (and reestimation of parameters).

setting $\beta_{XX} = \beta_{rr} = \beta_{\hat{\delta}\hat{\delta}} = \beta_{Xr} = \beta_{X\hat{\delta}} = 0$). We note that in both cases the return on the commodity in our model of Propositions 2 and 3 is Gaussian, which is consistent with the usual assumptions of the VAR framework. Figure 7 shows the 5-year holding period return distributions corresponding to the two sets of parameters and graphs the corresponding 5% VAR. The figure clearly shows that accounting for time variation in risk premia has a substantial impact on the dispersion of the holding period return, especially for copper, gold, and silver. The return distribution is more spread out for the case with a constant risk premia. Consequently, the VAR (the potential loss corresponding to a 5% tail event) for gold and silver more than doubles when the distribution is estimated without accounting for time variation in risk premia. This suggests that economic capital required to cover holdings in precious metals is significantly reduced when appropriately taking into account the dynamics of risk premia. The same figure shows that the VAR for oil is less significantly affected.

## V. Conclusions

We develop a three-factor model of commodity spot prices, convenience yields, and interest rates, which extends previous research in two ways. First, the model nests several (e.g., Brennan (1991), Gibson and Schwartz (1990), Schwartz (1997), Ross (1997), and Schwartz and Smith (2000)) proposed specifications. Second, it allows for time-varying risk premia. We show that previous models have implicitly imposed unnecessary restrictions on the unconditional correlation structure of commodity prices, convenience yields, and interest rates. In particular, the present model allows for convenience yields to be a function of spot commodity prices, which leads to mean reversion in spot prices. Mean reversion in spot prices can also be generated by negative correlation between risk premia and spot prices. The former affects the risk-neutral dynamics of commodity prices, that is, the cross section of futures prices. The latter affects only the historical measure dynamics of prices, that is, the time series of futures prices. Both components can thus be identified with panel data on futures prices. Using data on crude oil, copper, gold, and silver commodity futures, we empirically estimate the model using maximum likelihood. We find both features of the model to be economically and empirically significant. In particular, we find strong evidence for spot price level dependence in convenience yields of crude oil and copper, which implies mean reversion in spot prices under the risk-neutral measure, and which is consistent with the theory of storage. We find evidence for time-varying risk premia, which implies mean reversion of commodity prices under the physical measure, albeit with different strength and long-term mean. We also document the presence of a jump component in commodity prices.

The results suggest that the relative contribution of both effects to mean reversion (level-dependent convenience yield vs. time-varying risk premia) depends on the nature of the commodity, and, in particular, on the extent to which the commodity may serve as an input to production (e.g., a consumption good) versus a store of value (e.g., a financial asset). We find that for metals such as gold and silver, negative correlation between risk premia and spot prices explains most of the mean reversion, whereas for oil and copper, some of the mean reversion in spot prices is attributable also to convenience yields.[43] The analysis of various examples suggests that disentangling the sources of mean reversion and careful modeling of the dynamics of the convenience yield can have a substantial impact on (real) options valuation, investment decisions, and risk management.

## Appendix A. Closed-Form Solution for Futures Prices

The value of a futures contract with maturity $\tau$ is given by

$$F(Y,\tau) = \exp\left[A_F(\tau) + B_F(\tau)^T Y\right], \tag{A1}$$

where the closed-form solution for $A_F(\tau)$ and $B_F(\tau)$ is

[43] Of course, it is not clear that the primary use of gold and silver is as a store of value. Indeed, both have many industrial uses. Further theoretical research seems warranted to better understand these cross-sectional differences across commodities. Casassus et al. (2003) provide a first step in that direction.

$$
\begin{aligned}
A_F(\tau) = \phi_0 &+ \frac{1}{2}M_{11}^2\frac{1-e^{-2\tau\kappa_{11}^Q}}{2\kappa_{11}^Q} + \frac{1}{2}\left(M_{12}^2 + M_{22}^2\right)\frac{1-e^{-2\tau\kappa_{22}^Q}}{2\kappa_{22}^Q} \\
&+ \frac{1}{2}\left(M_{13}^2 + M_{23}^2 + M_{33}^2\right)\frac{1-e^{-2\tau\kappa_{33}^Q}}{2\kappa_{33}^Q} \\
&+ M_{11}M_{12}\frac{1-e^{-\tau\left(\kappa_{11}^Q+\kappa_{22}^Q\right)}}{\kappa_{11}^Q+\kappa_{22}^Q} + M_{11}M_{13}\frac{1-e^{-\tau\left(\kappa_{11}^Q+\kappa_{33}^Q\right)}}{\kappa_{11}^Q+\kappa_{33}^Q} \\
&+ \left(M_{12}M_{13} + M_{22}M_{23}\right)\frac{1-e^{-\tau\left(\kappa_{22}^Q+\kappa_{33}^Q\right)}}{\kappa_{22}^Q+\kappa_{33}^Q}
\end{aligned}
\tag{A2}
$$

and

$$
B_F(\tau) = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & M_{33} \end{pmatrix}\begin{pmatrix} e^{-\tau\kappa_{11}^Q} \\ e^{-\tau\kappa_{22}^Q} \\ e^{-\tau\kappa_{33}^Q} \end{pmatrix}
\tag{A3}
$$

with

$$M_{11} = \phi_1 + \alpha_1(\phi_2 + \alpha_2\phi_3) + \alpha_3\phi_3 \tag{A4}$$

$$M_{12} = -\alpha_1(\phi_2 + \alpha_2\phi_3) \tag{A5}$$

$$M_{13} = -\alpha_3\phi_3 \tag{A6}$$

$$M_{22} = \phi_2 + \alpha_2\phi_3 \tag{A7}$$

$$M_{23} = -\alpha_2\phi_3 \tag{A8}$$

$$M_{33} = \phi_3 \tag{A9}$$

and

$$\alpha_1 = \frac{\kappa_{21}^Q}{\kappa_{11}^Q - \kappa_{22}^Q} \tag{A10}$$

$$\alpha_2 = \frac{\kappa_{32}^Q}{\kappa_{22}^Q - \kappa_{33}^Q} \tag{A11}$$

$$\alpha_3 = \frac{\kappa_{31}^Q}{\kappa_{11}^Q - \kappa_{33}^Q} - \frac{\kappa_{21}^Q}{\kappa_{11}^Q - \kappa_{33}^Q}\frac{\kappa_{32}^Q}{\kappa_{22}^Q - \kappa_{33}^Q}. \tag{A12}$$

## Appendix B. Closed-Form Solution for Zero-Coupon Bonds

The value of a zero-coupon bond with maturity $\tau$ is given by

$$P(Y_1, \tau) = \exp[A_P(\tau) + B_P(\tau)Y_1], \tag{B1}$$

where the closed-form solution for $A_P(\tau)$ and $B_P(\tau)$ is

$$A_P(\tau) = -\left(\psi_0 - \frac{1}{2}\left(\frac{\psi_1}{\kappa_{11}^Q}\right)^2\right)\tau - \left(\frac{\psi_1}{\kappa_{11}^Q}\right)^2 \frac{1 - e^{-\tau\kappa_{11}^Q}}{\kappa_{11}^Q} + \frac{1}{2}\left(\frac{\psi_1}{\kappa_{11}^Q}\right)^2 \frac{1 - e^{-2\tau\kappa_{11}^Q}}{2\kappa_{11}^Q}, \tag{B2}$$

and

$$B_P(\tau) = -\psi_1 \frac{1 - e^{-\tau\kappa_{11}^Q}}{\kappa_{11}^Q}. \tag{B3}$$

## Appendix C. The $\{r, \delta, X\}$ Representation

We apply an invariant transformation to the canonical base to get the economic representation $\{r, \delta, X\}$ (see Dai and Singleton (2000)). This transformation rotates the state variables, but all the initial properties of the model are maintained, that is, the resulting model is a three-factor Gaussian model that is maximal. We have the transformations for $X(t)$, $r(t)$, and $\delta(t)$ from equations (1), (5), and (6), respectively.

$$\delta(t) = \eta_0 + \eta_Y^\top Y(t), \tag{C1}$$

where

$$\eta_0 = \psi_0 - \frac{1}{2}\phi_Y^\top \phi_Y, \tag{C2}$$

and

$$\eta_Y = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \psi_1 + \kappa_{11}^Q\phi_1 + \kappa_{21}^Q\phi_2 + \kappa_{31}^Q\phi_3 \\ \kappa_{22}^Q\phi_2 + \kappa_{32}^Q\phi_3 \\ \kappa_{33}^Q\phi_3 \end{pmatrix}. \tag{C3}$$

We define the transformed state vector $W^\top(t) = (r(t), \delta(t), X(t))$. The linear transformation in matrix form is

$$W(t) = \vartheta + LY(t), \tag{C4}$$

where $Y(t)$ follows the process in (2). The matrices for the linear transformations are

$$\vartheta = \begin{pmatrix} \psi_0 \\ \eta_0 \\ \phi_0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} \psi_1 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix}. \tag{C5}$$

From equation (C4) and Itô's lemma, we have

$$dW(t) = L\kappa^Q L^{-1}(\vartheta - W(t))\,dt + L\,dZ^Q(t). \tag{C6}$$

The mean reversion and long-run parameters under the equivalent martingale measure are given by $\kappa_r^Q = \kappa_{11}^Q$, $\kappa_{\delta r}^Q = -[L\kappa^Q L^{-1}]_{21}$, $\kappa_\delta^Q = -\kappa_{22}^Q - \kappa_{33}^Q$, $\kappa_{\delta X}^Q = \kappa_{22}^Q\kappa_{33}^Q$, $\theta_r^Q = \psi_0$, and $\kappa_{\delta 0}^Q = \eta_Y^\top \kappa^Q L^{-1}\vartheta$. Using the specification of the risk premia in equation (22) and equation (C4), we get the rotation under the physical measure as

$$\begin{aligned} dW(t) &= L\kappa^Q L^{-1}(\vartheta - W(t))\,dt + L\,dZ(t) + L(\beta_{0Y} + \beta_{1Y}Y(t))\,dt \\ &= L\kappa^Q L^{-1}(\vartheta - W(t))\,dt + L\,dZ(t) \\ &\quad + \left(L\beta_{0Y} - L\beta_{1Y}L^{-1}\vartheta + L\beta_{1Y}L^{-1}W(t)\right)dt. \end{aligned}$$

The risk-premia parameters for the $\{r, \delta, X\}$ representation are

$$\begin{pmatrix} \beta_{0r} \\ \beta_{0\delta} \\ \beta_{0X} \end{pmatrix} = L\beta_{0Y} - L\beta_{1Y}L^{-1}\vartheta \quad \text{and} \quad \begin{pmatrix} \beta_{rr} & \beta_{r\delta} & \beta_{rX} \\ \beta_{\delta r} & \beta_{\delta\delta} & \beta_{\delta X} \\ \beta_{Xr} & \beta_{X\delta} & \beta_{XX} \end{pmatrix} = L\beta_{1Y}L^{-1}. \tag{C7}$$

To get the covariance matrix, we match the instantaneous covariance matrices of the state variables from the model equation (C6) and the model in Proposition 1:

$$LL^T = \begin{pmatrix} \sigma_r^2 & \rho_{r\delta}\sigma_r\sigma_\delta & \rho_{rX}\sigma_r\sigma_X \\ \rho_{r\delta}\sigma_r\sigma_\delta & \sigma_\delta^2 & \rho_{\delta X}\sigma_\delta\sigma_X \\ \rho_{rX}\sigma_r\sigma_X & \rho_{\delta X}\sigma_\delta\sigma_X & \sigma_X^2 \end{pmatrix}. \tag{C8}$$

From equation (C8), we get that $\sigma_r^2 = \psi_1^2$, $\sigma_\delta^2 = \eta_Y^\top\eta_Y$, $\sigma_X^2 = \phi_Y^\top\phi_Y$, $\rho_{r\delta} = \frac{\eta_1}{\sigma_\delta}$, $\rho_{rX} = \frac{\phi_1}{\sigma_X}$, and $\rho_{\delta X} = \frac{\eta_Y^\top\phi_Y}{\sigma_\delta\sigma_X}$.

## Appendix D. The $\{r, \hat{\delta}, X\}$ Representation

We follow the same approach as in Appendix C. We have the transformations for $X(t)$ and $r(t)$ from equations (1) and (5), respectively. From equation (11) in Proposition 2 and equation (6) we can back out the idiosyncratic component of the convenience yield, $\hat{\delta}(t)$, as a function of the canonical state variables:[44]

$$\hat{\delta}(t) = \hat{\eta}_0 + \hat{\eta}_Y^\top Y(t), \tag{D1}$$

where

$$\hat{\eta}_0 = -\frac{1}{2}\phi_Y^\top\phi_Y + (1-\alpha_r)\psi_0 - \alpha_X\phi_0, \tag{D2}$$

[44] There are two possible decompositions for this representation that are equivalent to the ones in the proof of Proposition 2. We present the unique decomposition that satisfies the conditions of that proposition.

and

$$\hat{\eta}_Y = \begin{pmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \hat{\eta}_3 \end{pmatrix} = \begin{pmatrix} -\phi_2 \kappa_{21}^Q \frac{\kappa_{22}^Q - \kappa_{33}^Q}{\kappa_{11}^Q - \kappa_{22}^Q} - \phi_3 \frac{\kappa_{21}^Q \kappa_{32}^Q}{\kappa_{11}^Q - \kappa_{22}^Q} \\ \phi_2\left(\kappa_{22}^Q - \kappa_{33}^Q\right) + \phi_3 \kappa_{32}^Q \\ 0 \end{pmatrix}, \tag{D3}$$

with

$$\alpha_r = 1 + \frac{\phi_1\left(\kappa_{11}^Q - \kappa_{33}^Q\right) + \phi_2 \kappa_{21}^Q + \phi_3 \kappa_{31}^Q - \hat{\eta}_1}{\psi_1}, \tag{D4}$$

and

$$\alpha_X = \kappa_{33}^Q. \tag{D5}$$

We define the transformed state vector $\hat{W}^\top(t) = (r(t), \hat{\delta}(t), X(t))$. The linear transformation in matrix form is

$$\hat{W}(t) = \hat{\vartheta} + \hat{L}Y(t), \tag{D6}$$

where $Y(t)$ follows the process in (2). The matrices for the linear transformations are

$$\hat{\vartheta} = \begin{pmatrix} \psi_0 \\ \hat{\eta}_0 \\ \phi_0 \end{pmatrix} \quad \text{and} \quad \hat{L} = \begin{pmatrix} \psi_1 & 0 & 0 \\ \hat{\eta}_1 & \hat{\eta}_2 & \hat{\eta}_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix}. \tag{D7}$$

From equation (D6) and Itô's lemma we have

$$d\hat{W}(t) = \hat{L}\kappa^Q \hat{L}^{-1}(\hat{\vartheta} - \hat{W}(t))\,dt + \hat{L}\,dZ^Q(t). \tag{D8}$$

The remaining mean reversion and long-run parameters under the equivalent martingale measure are given by $\kappa_r^Q = \kappa_{11}^Q$, $\kappa_{\hat{\delta}}^Q = \kappa_{22}^Q$, $\theta_r^Q = \psi_0$, and $\theta_{\hat{\delta}}^Q = \hat{\eta}_0$. Using the specification of the risk premia in equation (22) and equation (D6), we get the rotation under the physical measure as

$$\begin{aligned} d\hat{W}(t) &= \hat{L}\kappa^Q \hat{L}^{-1}(\hat{\vartheta} - \hat{W}(t))\,dt + \hat{L}\,dZ(t) + \hat{L}(\beta_{0Y} + \beta_{1Y}Y(t))\,dt, \\ &= \hat{L}\kappa^Q \hat{L}^{-1}(\hat{\vartheta} - \hat{W}(t))\,dt + \hat{L}\,dZ(t) \\ &\quad + (\hat{L}\beta_{0Y} - \hat{L}\beta_{1Y}\hat{L}^{-1}\hat{\vartheta} + \hat{L}\beta_{1Y}\hat{L}^{-1}\hat{W}(t))\,dt. \end{aligned}$$

The risk-premia parameters for the $\{r, \hat{\delta}, X\}$ representation are

$$\begin{pmatrix} \beta_{0r} \\ \beta_{0\hat{\delta}} \\ \beta_{0X} \end{pmatrix} = \hat{L}\beta_{0Y} - \hat{L}\beta_{1Y}\hat{L}^{-1}\hat{\vartheta} \quad \text{and} \quad \begin{pmatrix} \beta_{rr} & \beta_{r\hat{\delta}} & \beta_{rX} \\ \beta_{\hat{\delta}r} & \beta_{\hat{\delta}\hat{\delta}} & \beta_{\hat{\delta}X} \\ \beta_{Xr} & \beta_{X\hat{\delta}} & \beta_{XX} \end{pmatrix} = \hat{L}\beta_{1Y}\hat{L}^{-1}. \tag{D9}$$

To get the covariance matrix, we match the instantaneous covariance matrices of the state variables from the model equation (D8) and the model in Proposition 2:

$$\hat{L}\hat{L}^T = \begin{pmatrix} \sigma_r^2 & \rho_{r\hat{\delta}}\sigma_r\sigma_{\hat{\delta}} & \rho_{rX}\sigma_r\sigma_X \\ \rho_{r\hat{\delta}}\sigma_r\sigma_{\hat{\delta}} & \sigma_{\hat{\delta}}^2 & \rho_{\hat{\delta}X}\sigma_{\hat{\delta}}\sigma_X \\ \rho_{rX}\sigma_r\sigma_X & \rho_{\hat{\delta}X}\sigma_{\hat{\delta}}\sigma_X & \sigma_X^2 \end{pmatrix}. \tag{D10}$$

From equation (D10), we get $\sigma_r^2 = \psi_1^2$, $\sigma_{\hat{\delta}}^2 = \hat{\eta}_Y^\top \hat{\eta}_Y$, $\sigma_X^2 = \phi_Y^\top \phi_Y$, $\rho_{r\hat{\delta}} = \frac{\hat{\eta}_1}{\sigma_{\hat{\delta}}}$, $\rho_{rX} = \frac{\phi_1}{\sigma_X}$, and $\rho_{\hat{\delta}X} = \frac{\hat{\eta}_Y^\top \phi_Y}{\sigma_{\hat{\delta}}\sigma_X}$.

## Appendix E. MLE of the Maximal Model

We follow the maximum-likelihood approach of Chen and Scott (1993) and Pearson and Sun (1994) with a slight modification: instead of assuming that we observe without error some futures contracts and bond prices, we follow Collin-Dufresne et al. (2002) and choose to fit the principal components (PCs) of futures and bonds.[45] From the perfectly observed data and using the closed-form solutions for futures and bonds in Appendices A and B, we can invert for the state variables $Y(t)$. We assume that the first two PCs of the futures curve and the first PC of the yield curve are observed without error. The rest of the PCs are assumed to be observed with measurement errors that are jointly normally distributed and follow an AR(1) process. Using the principal component approach instead of single contracts has some advantages. First, it guarantees (by construction) that we fit perfectly the first two PCs of futures and the first PC of the yield curve. Second, it orthogonalizes the matrix of measurement errors. Finally, the principal component approach dispenses the arbitrariness of what contracts are perfectly observed.

Our cross sectional data set is composed of $m$ futures contracts and $n$ zero-coupon bonds. The $i^{\text{th}}$ principal component of the futures curve is

$$PC_F^i(\hat{Y};\Theta) = {\omega_F^i}^\top \left(\overline{A_F}(\Theta) + \overline{B_F}(\Theta)^\top \hat{Y}\right), \tag{E1}$$

where $\omega_F^i$ is an $m \times 1$ eigenvector corresponding to the $i^{\text{th}}$ principal component, $\overline{A_F}(\Theta)$ is an $m \times 1$ vector, and $\overline{B_F}(\Theta)$ is a $3 \times m$ matrix that determines the theoretical value of the log futures prices for different maturity contracts, that is, $\overline{A_F}(\Theta)^\top = (A_F(\tau_F^1;\Theta), \ldots, A_F(\tau_F^m;\Theta))$ and $\overline{B_F}(\Theta) = (B_F(\tau_F^1;\Theta), \ldots, B_F(\tau_F^m;\Theta))$ (see Appendix A). Finally, $\Theta$ is the parameter space of the model.

In the same way, we obtain the $i^{\text{th}}$ principal component of the yield curve:

$$PC_P^i(\hat{Y}_1;\Theta) = {\omega_P^i}^\top (\overline{A_P}(\Theta) + \overline{B_P}(\Theta)\hat{Y}_1). \tag{E2}$$

[45] These principal components are linear in the state variables and can be thought of as being portfolios of single contracts.

To invert the state variables, $\hat{Y}$, we perfectly observe the first two principal components of the futures curve and the first principal component of the yield curve. The relation between the data and the state variables is

$$G(t) = A(\Theta) + B(\Theta)\hat{Y}(t), \tag{E3}$$

where $G(t)^\top = (\omega_P^{1\,\top}\overline{LnP}(t), \omega_F^{1\,\top}\overline{LnF}(t), \omega_F^{2\,\top}\overline{LnF}(t))$ is the $1 \times 3$ vector of perfectly observed PCs at time t,[46] $A(\Theta)^\top = (\omega_P^{1\,\top}\overline{A_P}(\Theta), \omega_F^{1\,\top}\overline{A_F}(\Theta), \omega_F^{2\,\top}\overline{A_F}(\Theta))$ is a $1 \times 3$ vector, and $B$ is a $3 \times 3$ matrix given by

$$B(\Theta) = \begin{pmatrix} \omega_P^{1\,\top}\overline{B_P}(\Theta) & 0 & 0 \\ & \omega_F^{1\,\top}\overline{B_F}(\Theta)^\top & \\ & \omega_F^{2\,\top}\overline{B_F}(\Theta)^\top & \end{pmatrix}. \tag{E4}$$

At any given point in time $t$, we can invert equation (E3) to back out the state variables $\hat{Y}(t)^\top = (\hat{Y}_1(t), \hat{Y}_2(t), \hat{Y}_3(t))$:

$$\hat{Y}(t) = B(\Theta)^{-1}(G(t) - A(\Theta)). \tag{E5}$$

The other bonds and futures principal components are priced with error:

$$\omega_P^{i\,\top}\overline{LnP}(t) = \omega_P^{i\,\top}(\overline{A_P}(\Theta) + \overline{B_P}(\Theta)\hat{Y}_1(t)) + u_P^i(t) \quad \text{for } i = 2, \ldots, n, \tag{E6}$$

$$\omega_F^{i\,\top}\overline{LnF}(t) = \omega_F^{i\,\top}(\overline{A_F}(\Theta) + \overline{B_F}(\Theta)^\top\hat{Y}(t)) + u_F^i(t) \quad \text{for } i = 3, \ldots, m. \tag{E7}$$

We assume that the measurement errors $u_j^i(t)$ follow an AR(1) process, that is, $u_j^i(t) = \rho_j u_j^i(t-1) + e_j^i(t)$ for $j \in \{P, F\}$, and the errors $e_j^i(t)$ are jointly normally distributed with 0 mean and covariance matrix $E[e_j^i e_j^{i\,\top}]$.

The conditional likelihood function for every time $t$ will be given by the likelihood function of $G(t)$ times the likelihood function of the measurement errors, $f_u(u(t) \mid u(t-1)) = f_e(e(t))$. We do not know the conditional density function of $G(t)$, but since it is an affine function of the state vector $\hat{Y}(t)$, and given we know the conditional distribution of $\hat{Y}(t)$,[47] we can get it using the relation in equation (E5):

$$f_G(G(t) \mid G(t-1)) = \text{abs}(J_{\hat{Y}}) f_{\hat{Y}}(\hat{Y}(t) \mid \hat{Y}(t-1)), \tag{E8}$$

where $J_{\hat{Y}}$ is the Jacobian of the transformation from $G(t)$ to $\hat{Y}(t)$, that is, $J_{\hat{Y}} = \det(B^{-1})$.

The estimated parameters will be the ones that maximize the log-likelihood function

[46] Note that $\overline{LnP}(t)$ and $\overline{LnF}(t)$ are the vectors of the logarithm of the observed bonds and futures contracts at time $t$, respectively.

[47] In our Gaussian model, we can calculate the exact moments for the distribution of $\hat{Y}(t)$.

$$\max_{\Theta} L(\Theta) = \sum_{t=1}^{T} f_G(G(t) \mid G(t-1)) + f_e(e(t)), \tag{E9}$$

where $f_G(G(1) \mid G(0))$ is the unconditional density function.

## Appendix F. Closed-Form Solution for Futures Prices with Jumps

The model is given by

$$\delta(t) = \alpha_r r(t) + \hat{\delta}(t) + \alpha_X X(t), \tag{F1}$$

where the state variables $\{r, \hat{\delta}, X\}$ have the following risk-neutral dynamics:

$$dr(t) = \kappa_r^Q \left(\theta_r^Q - r(t)\right) dt + \sigma_r \, dZ_r^Q(t), \tag{F2}$$

$$d\hat{\delta}(t) = \kappa_{\hat{\delta}}^Q \left(\theta_{\hat{\delta}}^Q - \hat{\delta}(t)\right) dt + \sigma_{\hat{\delta}} \, dZ_{\hat{\delta}}^Q(t), \tag{F3}$$

and

$$dX(t) = \left(r(t) - \delta(t) - \frac{1}{2}\sigma_X^2 - \sum_{i=1}^{3}(\varphi_i - 1)\lambda_i^Q\right) dt + \sigma_X \, dZ_X^Q(t) + \sum_{i=1}^{3} \upsilon_i(t) \, dN_i(t). \tag{F4}$$

The futures price is given by $F^T(t) = E_t^Q[S(T)] = E_t^Q[e^{X(T)}]$. We show that the expectation has the following exponential affine form:

$$F^T(t) = \exp(A_0(T-t) + B_X(T-t)X(t) + B_r(T-t)r(t) + B_{\hat{\delta}}(T-t)\hat{\delta}(t)), \tag{F5}$$

where the functions $A_0, B_X, B_r,$ and $B_{\hat{\delta}}$ are given by

$$B_X(\tau) = e^{-\alpha_X \tau}, \tag{F6}$$

$$B_{\hat{\delta}}(\tau) = \frac{1}{\alpha_X - \kappa_{\hat{\delta}}}\left(e^{-\alpha_X \tau} - e^{-\kappa_{\hat{\delta}} \tau}\right), \tag{F7}$$

$$B_r(\tau) = \frac{\alpha_r - 1}{\alpha_X - \kappa_r}\left(e^{-\alpha_X \tau} - e^{-\kappa_r \tau}\right), \tag{F8}$$

and

$$A_0(\tau) = \int_0^\tau \left\{ \frac{1}{2}\left(B_X^2(s)\sigma_X^2 + B_{\hat{\delta}}^2(s)\sigma_{\hat{\delta}}^2 + B_r^2(s)\sigma_r^2\right) - \sum_i \lambda_i (B_X(s)(\varphi_i - 1) - (\varphi_i(B_X(s)) - 1)) + B_{\hat{\delta}}(s)\kappa_{\hat{\delta}}\theta_{\hat{\delta}} + B_r(s)\kappa_r\theta_r + \rho_{r\hat{\delta}}\sigma_r\sigma_{\hat{\delta}}B_r(s)B_{\hat{\delta}}(s) + \rho_{rX}\sigma_r\sigma_X B_r(s)B_X(s) + \rho_{\hat{\delta}X}\sigma_X\sigma_{\hat{\delta}}B_X(s)B_{\hat{\delta}}(s) \right\} ds, \tag{F9}$$

where we define $\varphi_i(\alpha) = \exp(\alpha m_i + \frac{\alpha^2\sigma_i^2}{2})$. The proof consists in verifying that the candidate solution given in equation (F5) is a $Q$-martingale. Indeed, applying Itô's lemma to $F$ defined in (F5) and using equations (F6)–(F9), we see that

$$\begin{aligned} F^T(t) = F^T(T) - \int_t^T &\big\{B_X(T-s)\sigma_X\, dZ_X(s) + B_{\hat\delta}(T-s)\sigma_{\hat\delta}\, dZ_{\hat\delta}(s) \\ &+ B_r(T-s)\sigma_r\, dZ_r(s)\big\} \\ &+ \sum_i \int_t^T \big\{\big(e^{B_X(s)\upsilon_i(s)} - 1\big)\, dN_i(s) - (\varphi_i(B_X(s)) - 1)\lambda_i^Q ds\big\}. \end{aligned} \tag{F10}$$

Thus,

$$F^T(t) = E_t^Q[F^T(T)] = E_t^Q\big[e^{X(T)}\big],$$

where for the second equality we use the fact that $B_X(0) = 1$ and $A_0(0) = B_{\hat\delta}(0) = B_r(0) = 0$.

Inspecting the solution we see that the only impact of jumps on the prices of futures is through the term

$$J_i := \lambda_i^Q(B_X(s)(\varphi_i - 1) - \varphi_i(B_X(s)) - 1) \tag{F11}$$

in the expression for $A_0$.

Note that if $\alpha_X = 0$, then $B_X(s) = 1$ and thus we have $J_i = 0$. We conclude that if $\alpha_X = 0$, then futures prices are not affected by the presence of jumps.

Next we show that if $\alpha_X \neq 0$, then the impact of jumps is likely to be small if the jump intensity and jump size volatility are "small." Indeed, if a Taylor-series approximation is appropriate, we have $\varphi_i \approx 1 + (m_i + \frac{\sigma_i^2}{2})$ and $\varphi_i(B_X) \approx 1 + (m_i B_X + \frac{v_i^2 B_X^2}{2})$. Substituting in the expression for $J_i$ we obtain

$$J_i \approx \lambda_i^Q \frac{v_i^2}{2} B_X(1 - B_X). \tag{F12}$$

We thus see that the impact of jumps for the cross section of futures prices is minimal if the jump intensity and jump size volatility are small. Thus, jumps are mainly helpful in capturing time-series properties of futures prices. Of course, jumps would have a significant impact for the cross section of option prices.

## Appendix G. MLE of the Triple-Jump Model

For the case with a triple-jump component in spot prices we use an approach similar to the one in Appendix E. Since the jumps are in the spot price, it is easier to work with the economic representation $\{r, \hat\delta, X\}$ than with the canonical form $\{Y_1, Y_2, Y_3\}$. Using both equation (C4) from Appendix D and equation (E5), the conditional likelihood of the perfectly observed principal components is

$$f_G(G(t)\,|\,G(t-1)) = \mathrm{abs}(J_{\hat W}) f_{\hat W}(\hat W(t)\,|\,\hat W(t-1)), \tag{G13}$$

where $\hat{W}$ is the vector of the economic state variables $\{r, \hat{\delta}, X\}$ implied from the perfectly observed data and $J_{\hat{W}}$ is the Jacobian of the transformation from $G(t)$ to $\hat{W}(t)$. Equation (G13) is similar to equation (E8), but depends on the economic state variables $\hat{W}$ instead of the canonical vector $\hat{Y}$. We also use the closed-form solution for futures prices with jumps from Appendix F instead of the one in Appendix A.

Since the transition density with jumps is no longer Gaussian, we follow Ball and Torous (1983), Jorion (1988), and Das (2002) and approximate it by a mixture of Gaussians. This approximation is as follows:

$$
\begin{aligned}
& f_{\hat{W}}(\hat{W}(t) \mid \hat{W}(t-1)) \\
&= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} p_1(N_1(\Delta t) = k_1) p_2(N_2(\Delta t) = k_2) p_3(N_3(\Delta t) = k_3) \\
&\quad \times f_{\hat{W}}(\hat{W}(t) \mid \hat{W}(t-1), N_1(\Delta t) = k_1, N_2(\Delta t) = k_2, N_3(\Delta t) = k_3),
\end{aligned}
$$

where the last term is the likelihood function and is Gaussian conditional on a fixed number of jumps $k_1, k_2$, and $k_3$. The Poisson probabilities are

$$p_i(N_i(\Delta t) = k_i) = e^{-\lambda_i \Delta t} \frac{(\lambda_i \Delta t)^{k_i}}{k_i!}. \tag{G14}$$

This approximation would be exact if the time interval were infinitesimal.

## REFERENCES


Amin, Kaushik, Victor Ng, and Craig Pirrong, 1995, Valuing energy derivatives, in Robert Jameson, eds.: *Managing Energy Price Risk* (Risk Publications, London).

Bakshi, Gurdip, and Dilip Madan, 2000, Spanning and derivative security valuation, *Journal of Financial Economics* 55, 205–238.

Ball, Clifford A., and Walter N. Torous, 1983, A simplified jump process for common stock returns, *Journal of Financial and Quantitative Analysis* 18, 53–65.

Bessembinder, Hendrik, Jay F. Coughenour, Paul J. Seguin, and Margaret Monroe Smoller, 1995, Mean-reversion in equilibrium asset prices: Evidence from the futures term structure, *Journal of Finance* 50, 361–375.

Brémaud, Pierre, 1981, *Point Processes and Queues: Martingale Dynamics* (Springer-Verlag, New York).

Brennan, Michael J., 1958, The supply of storage, *American Economic Review* 48, 50–72.

Brennan, Michael J., 1991, The price of convenience and the valuation of commodity contingent claims, in Diderik Lund, and Bernt Øksendal, eds.: *Stochastic Models and Option Values* (North-Holland, Amsterdam).

Brennan, Michael J., and Eduardo S. Schwartz, 1985, Evaluating natural resource investments, *Journal of Business* 58, 135–157.

Casassus, Jaime, Pierre Collin-Dufresne, and Bryan R. Routledge, 2003, Equilibrium commodity prices with irreversible investment and non-linear technologies, Working paper, Carnegie Mellon University.

Chen, Ren R., and Louis Scott, 1993, Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates, *Journal of Fixed Income* 3, 14–31.

Collin-Dufresne, Pierre, and Robert S. Goldstein, 2002, Do bonds span the fixed income markets? Theory and evidence for unspanned stochastic volatility, *Journal of Finance* 57, 1685–1730.

Collin-Dufresne, Pierre, Robert S. Goldstein, and Christopher S. Jones, 2002, Identification and estimation of 'maximal' affine term structure models: An application to stochastic volatility, Working paper, Carnegie Mellon University.

Collin-Dufresne, Pierre, and Julien N. Hugonnier, 1999, On the pricing and hedging of contingent claims in the presence of extraneous risks, Working paper, Carnegie Mellon University.

Collin-Dufresne, Pierre, and Bruno Solnik, 2001, On the term structure of default premia in the swap and libor markets, *Journal of Finance* 56, 1095–1115.

Dai, Qiang, and Kenneth J. Singleton, 2000, Specification analysis of affine term structure models, *Journal of Finance* 55, 1943–1978.

Dai, Qiang, and Kenneth J. Singleton, 2002, Expectation puzzles, time-varying risk premia, and affine models of the term structure, *Journal of Financial Economics* 63, 415–441.

Das, Sanjiv R., 2002, The surprise element: Jumps in interest rates, *Journal of Econometrics* 106, 27–65.

Deaton, Angus, and Guy Laroque, 1992, On the behavior of commodity prices, *Review of Economic Studies* 59, 1–23.

Duffee, Gregory R., 2002, Term premia and interest rate forecasts in affine models, *Journal of Finance* 57, 405–443.

Duffee, Gregory, and Richard Stanton, 2002, Estimation of dynamic term structure models, Working paper, UC Berkeley.

Duffie, Darrell, 1996, *Dynamic Asset Pricing Theory* (Princeton University Press, Princeton).

Duffie, Darrell, and Rui Kan, 1996, A yield-factor model of interest rates, *Mathematical Finance* 6, 379–406.

Duffie, Darrell, Jun Pan, and Kenneth J. Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, 1343–1376.

Duffie, Darrell, and Kenneth J. Singleton, 1997, An econometric model of the term structure of interest-rate swap yields, *Journal of Finance* 52, 1287–1321.

Fama, Eugene F., 1984, Forward and spot exchange rates, *Journal of Monetary Economics* 14, 319–338.

Fama, Eugene F., and Kenneth R. French, 1987, Commodity futures prices: Some evidence on forecast power, premiums and the theory of storage, *Journal of Business* 60, 55–73.

Fama, Eugene F., and Kenneth R. French, 1988, Permanent and temporary components of stock prices, *Journal of Political Economy* 96, 246–273.

Gibson, Rajna, and Eduardo S. Schwartz, 1990, Stochastic convenience yield and the pricing of oil contingent claims, *Journal of Finance* 45, 959–976.

Harrison, J. Michael, and David M. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* 20, 381–408.

Heston, Steven L., 1993, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* 6, 327–343.

Hilliard, Jimmy E., and Jorge Reis, 1998, Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot, *Journal of Financial and Quantitative Analysis* 33, 61–86.

Honoré, Peter, 1998, Pitfalls in estimating jump-diffusion models, Working paper, Aarhus University.

Hull, John, 2000, *Options, Futures and Other Derivative Securities* (Prentice Hall, New Jersey).

Jameson, Robert, ed., 1995, *Managing Energy Price Risk* (Risk Publications, London).

Jarrow, Robert A., David Lando, and Fan Yu, 2005, Default risk and diversification: Theory and empirical implications, *Mathematical Finance* 15, 1–26.

Jorion, Philippe, 1988, On jump processes in the foreign exchange and stock markets, *Review of Financial Studies* 1, 427–445.

Langetieg, Terence C., 1980, A multivariate model of the term structure, *Journal of Finance* 35, 71–97.

Liptser, Robert S., and Albert N. Shiryaev, 1977, *Statistics of Random Processes* (Springer-Verlag, New York).

Litterman, Robert, and José A. Scheinkman, 1991, Common factors affecting bond returns, *Journal of Fixed Income* 1, 54–61.

Litzenberger, Robert H., and Nir Rabinowitz, 1995, Backwardation in oil futures markets: Theory and empirical evidence, *Journal of Finance* 50, 1517–1545.

Majd, Saman, and Robert S. Pindyck, 1987, Time to build, option value, and investment decisions, *Journal of Financial Economics* 18, 7–27.

Pearson, Neil D., and Tong-Sheng Sun, 1994, Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll and Ross model, *Journal of Finance* 49, 1279–1304.

Piazzesi, Monika, 2005, Affine term structure models, in Yacine Ait-Sahalia, and Lars P. Hansen, eds.: *Handbook of Financial Econometrics* (North-Holland, Amsterdam, forthcoming).

Ross, Stephen A., 1997, Hedging long run commitments: Exercises in incomplete market pricing, *Economic Notes by Banca Monte* 26, 99–132.

Routledge, Bryan R., Duane J. Seppi, and Chester S. Spatt, 2000, Equilibrium forward curves for commodities, *Journal of Finance* 55, 1297–1338.

Schwartz, Eduardo S., 1997, The stochastic behavior of commodity prices: Implications for valuation and hedging, *Journal of Finance* 52, 923–973.

Schwartz, Eduardo S., and James E. Smith, 2000, Short-term variations and long-term dynamics in commodity prices, *Management Science* 46, 893–911.

Seppi, Duane J., 2003, Risk-neutral stochastic processes for commodity derivative pricing: An introduction and survey, in Ehud I. Ronn, ed.: *Real Options and Energy Management Using Options Methodology to Enhance Capital Budgeting Decisions* (Risk Publications, London).

Vasicek, Oldrich, 1977, An equilibrium characterization of the term structure, *Journal of Financial Economics* 5, 177–188.

Working, Holbrook, 1949, The theory of price of storage, *American Economic Review* 39, 1254–1262.